Differences of Functions with the Same Value Set

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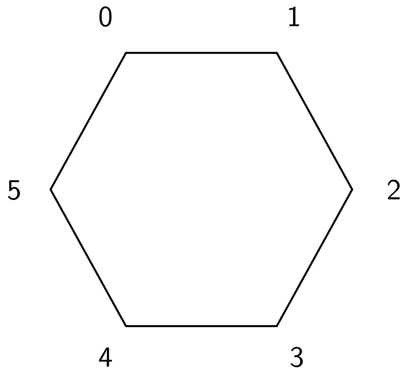
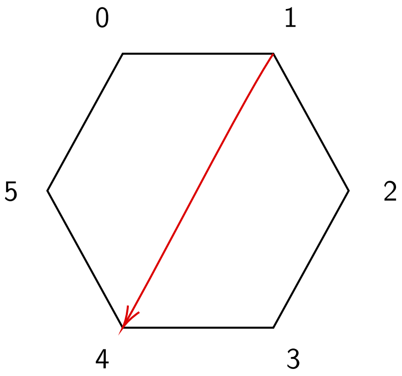
SEICCGTC
2021

Outline

A Geometry Problem



A Geometry Problem



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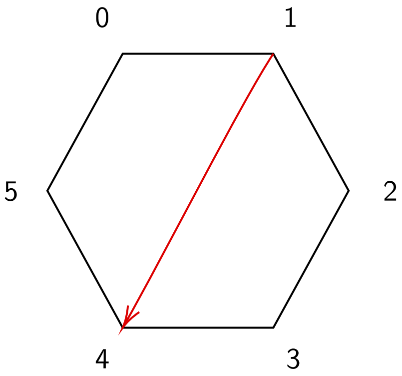

$\mathbf{b} = 4$
$\mathbf{c} = 1$
$\mathbf{a} = 3$

A Geometry Problem

$$\mathbf{a} = 0, 1, 2, 2, 3, 4$$

A Geometry Problem

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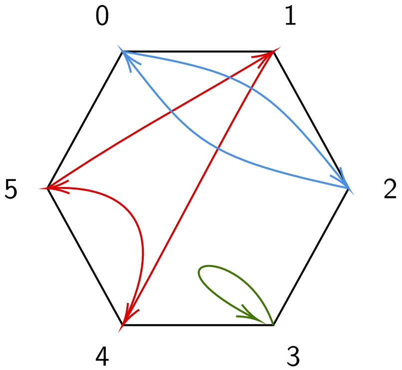

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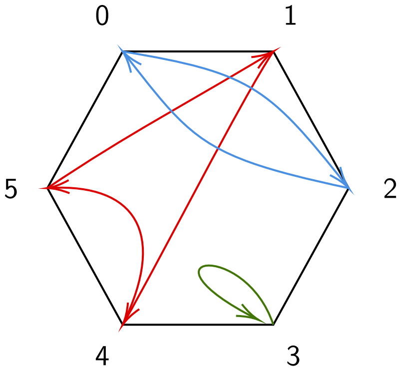

$$\mathbf{b} = 3, 5, 2, 1, 4, 0$$
$$\mathbf{c} = 3, 4, 0, 5, 1, 2$$
$$\mathbf{a} = 0, 1, 2, 2, 3, 4$$

An Algebraic Perspective

Algebraic Formulation

$a : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ When can we write $a(i) \equiv b(i) - c(i)$ *(mod n)* where $b, c : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ bijections?

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Hall's Theorem

Let a be a function $a : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$. Then $a = b - c$ with b, c bijections on $\mathbb{Z}_n$ if and only if $\sum_{i \in \mathbb{Z}_n} a(i) \equiv 0$ *(mod n)*.

A small tweak

Question

What if we only specify that the indegree and outdegree of each vertex are the same?

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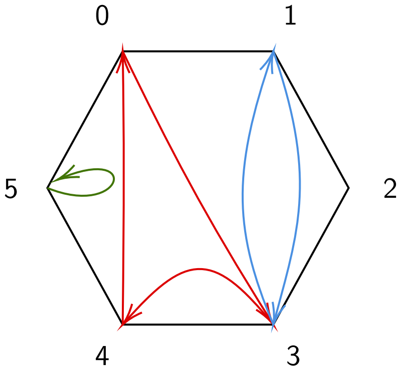

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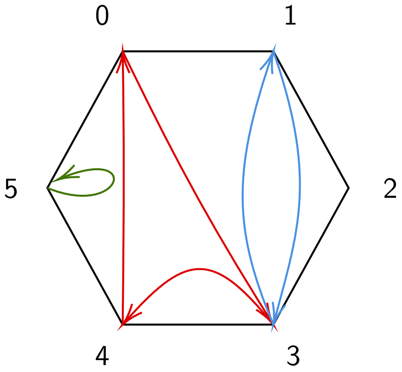

$$\mathbf{b} = 5, 4, 3, 0, 3, 1$$
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Our Problem

Problem

Given a function $a : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ and a multiset $M = \{\{g_1, \ldots, g_n\}\}$ of elements in $\mathbb{Z}_n$, can we write $a = b - c$ for two functions $b, c : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ such that $\{\{b(i) \mid i = 1, \ldots, n\}\} = \{\{c(i) \mid i = 1, \ldots, n\}\} = M$?

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Problem

Given a sequence $\mathbf{a} = a_1, \ldots, a_k$ of elements in $\mathbb{Z}_n$ and multiset M. When can we write $\mathbf{a} = \mathbf{b} - \mathbf{c}$ two sequences $\mathbf{b} = b_1, \ldots, b_k$, $\mathbf{c} = c_1, \ldots, c_k$ such that $\mathbf{b}, \mathbf{c}$, are reorderings of M?

A Useful Definition

Definition

Let $\mathbf{a} = a_1, \ldots, a_k$ be a sequence of elements in $\mathbb{Z}_n$. We define a new sequence $\mathbf{S}(\mathbf{a}) = S(a_1), S(a_2), \ldots, S(a_k)$ where $S(a_j) = \sum_{i=1}^{j} a_i$. Let $l \in \mathbb{Z}_n$, we denote $l + \mathbf{S}(\mathbf{a}) := l + S(a_1), l + S(a_2), \ldots, l + S(a_k)$

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Example

$\mathbf{a} = 1, 3, 3, 4, 6, 1, 3,\ a_i \in \mathbb{Z}_7$.

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$\mathbf{S}(a_1) = 1$

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$\vdots$

$\mathbf{S}(a_7) = 1 + 3 + 3 + 6 + 1 + 3 \equiv 0 \pmod 7$.

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$\mathbf{S}(\mathbf{a}) = 1, 4, 0, 4, 3, 4, 0$

First Result

Definition

Let $\mathbf{a} = a_1, \ldots, a_k$ be a sequence of elements in $\mathbb{Z}_n$. Then $\mathbf{a}$ is a **zero sum sequence** if $\sum_{i=1}^{k} a_i \equiv 0$ *(mod n)*.

First Result

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Let $\mathbf{a} = a_1, \ldots, a_k$ be a sequence of elements in $\mathbb{Z}_n$. Then $\mathbf{a}$ is a **zero sum sequence** if $\sum_{i=1}^{k} a_i \equiv 0$ *(mod n)*.

Proposition

Let $\mathbf{a} = a_1, \ldots, a_k$ be a sequence of elements in $\mathbb{Z}_n$. There exist sequences $\mathbf{b} = b_1, \ldots, b_k$, $\mathbf{c} = c_1, \ldots, c_k$, where $\mathbf{c}$ is a rearrangement of $\mathbf{b}$ with $a_i = b_i - c_i$ if and only if $\sum_{i=1}^{k} a_i \equiv 0 \pmod{n}$.

Example

Example

$\mathbf{a} = 3, 2, 5, 4, 1, 1, 5, \; a_i \in \mathbb{Z}_7.$

Example

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$\mathbf{a} = 3, 2, 5, 4, 1, 1, 5,\ a_i \in \mathbb{Z}_7$. Let $\mathbf{b} = \mathbf{S}(\mathbf{a}) = 3, 5, 3, 0, 1, 2, 0.$

Example

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$\mathbf{a} = 3, 2, 5, 4, 1, 1, 5,\ a_i \in \mathbb{Z}_7$. Let $\mathbf{b} = \mathbf{S}(\mathbf{a}) = 3, 5, 3, 0, 1, 2, 0$.

$$\mathbf{b} : 3 \quad 5 \quad 3 \quad 0 \quad 1 \quad 2 \quad 0$$

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$$\begin{array}{rl} \mathbf{b}: & 3 \quad 5 \quad 3 \quad 0 \quad 1 \quad 2 \quad 0 \\ \mathbf{c}: & 0 \quad 3 \quad 5 \quad 3 \quad 0 \quad 1 \quad 2 \\ \hline \end{array}$$

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Remark

$S(a_k) \equiv 0$ *(mod n)* if and only if **a** is zero sum.

Example

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Definition

Let $\mathbf{a} = a_1, \ldots, a_k$ be a zero sum sequence. We say $\mathbf{a}$ is irreducible if $\mathbf{S}(a_m) \neq 0$ for $m < k$. Otherwise $\mathbf{a}$ is reducible.

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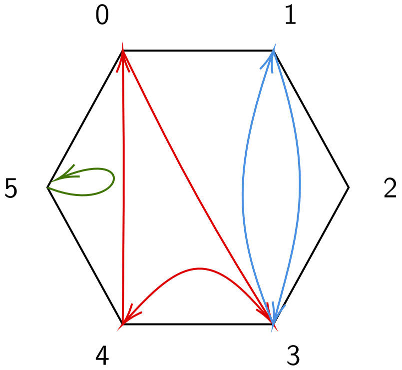
Remark

Note that $\mathbf{a} = \mathbf{a}_1 * \mathbf{a}_2$ where $*$ is concatenation and $\mathbf{a}_1 = 3, 2, 5, 4$, $\mathbf{a}_2 = 1, 1, 5$. Note that both $\mathbf{a}_1$ and $\mathbf{a}_2$ are zero sum irreducible.

An Illustrative Illustration



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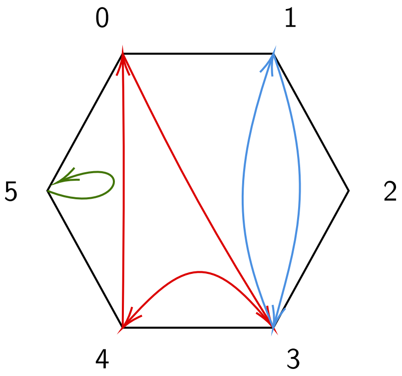

$$\mathbf{b} = 5, 4, 0, 3, 3, 0$$
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Notation

Let $\mathbf{x} = x_1, \ldots, x_k$ be a sequence of elements and $\sigma \in \mathbb{S}_k$. Define $\sigma(\mathbf{x}) = x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}$.

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Theorem

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Theorem

Let $\mathbf{a} = a_1, \ldots, a_k$ *be a zero sum sequence of elements in* $\mathbb{Z}_n$*. Let* $\mathbf{b} = b_1, \ldots, b_k$ *and* $\mathbf{c} = c_1, \ldots, c_k$ *be rearrangements of each other.*

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Let $\mathbf{x} = x_1, \ldots, x_k$ be a sequence of elements and $\sigma \in \mathbb{S}_k$. Define $\sigma(\mathbf{x}) = x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}$.

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Let $\mathbf{a} = a_1, \ldots, a_k$ *be a zero sum sequence of elements in* $\mathbb{Z}_n$. *Let* $\mathbf{b} = b_1, \ldots, b_k$ *and* $\mathbf{c} = c_1, \ldots, c_k$ *be rearrangements of each other. Then* $\mathbf{a} = \mathbf{b} - \mathbf{c}$ *if and only if there exists* $\sigma \in \mathbb{S}_k$ *such that* $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$ *,* $\sigma(\mathbf{a}) = \sigma(\mathbf{a}_1) * \cdots * \sigma(\mathbf{a}_m)$ *where each* $\sigma(\mathbf{a}_i)$ *is zero sum irreducible and* $\sigma(\mathbf{b}) = \sigma(\mathbf{b}_1) * \cdots * \sigma(\mathbf{b}_m)$, $\sigma(\mathbf{c}) = \sigma(\mathbf{c}_1) * \cdots * \sigma(\mathbf{c}_m)$, $1 \leq m \leq k$, *where each* $\sigma(\mathbf{c}_i)$ *is* 1*-right shift of* $\sigma(\mathbf{b}_i)$ *and each* $\sigma(\mathbf{a}_i)$ *is zero sum irreducible.*

Lemma

Let $\mathbf{a}$ *be a zero sum sequence. Let* $\mathbf{b}$ *and* $\mathbf{c}$ *be sequences such that* $\mathbf{a} = \mathbf{b} - \mathbf{c}$ *and* $\mathbf{c}$ *is a reordering of* $\mathbf{b}$*. Then* $\mathbf{c}$ *is* 1*-right shift of* $\mathbf{b}$ *if and only if* $\mathbf{b} = l + \mathbf{S}(\mathbf{a})$ *for some* $l \in \mathbb{Z}_n$*.*

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Example

$\mathbf{a} = 1, 2, 2, 1,\ a_i \in \mathbb{Z}_6.$

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Example

$\mathbf{a} = 1, 2, 2, 1,\ a_i \in \mathbb{Z}_6$. Let $\mathbf{b} = \mathbf{S}(\mathbf{a}) = 1, 3, 5, 0$ and $\mathbf{c} = 0, 1, 3, 5$.

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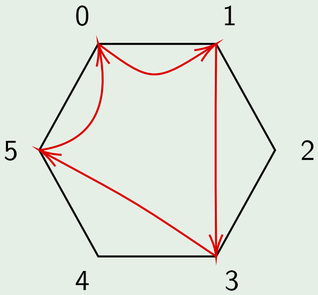
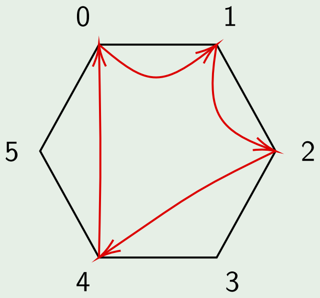


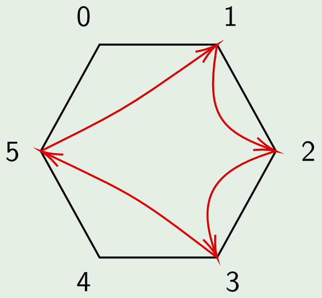
Example

Example

$\mathbf{b} = 1 + \mathbf{S}(\mathbf{a}) = 2, 4, 0, 1$



$\mathbf{b} = 2 + \mathbf{S}(\mathbf{a}) = 3, 5, 1, 2$



Putting Everything Together

What We Know

- $\mathbf{a} = \mathbf{b} - \mathbf{c}$ iff $\exists\ \sigma \in \mathbb{S}_k$ such that $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$ where $\sigma(\mathbf{a}) = \sigma(\mathbf{a}_1) * \cdots * \sigma(\mathbf{a}_m)$ with $\sigma(\mathbf{a}_i)$ zero sum irred and $\sigma(\mathbf{b}) = \sigma(\mathbf{b}_1) * \cdots * \sigma(\mathbf{b}_m)$, $\sigma(\mathbf{c}) = \sigma(\mathbf{c}_1) * \cdots * \sigma(\mathbf{c}_m)$ where $\sigma(\mathbf{c}_i)$ 1-right shift of $\sigma(\mathbf{b}_i)$

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- $\mathbf{a} = \mathbf{b} - \mathbf{c}$. Then $\mathbf{c}$ is 1-right shift of $\mathbf{b}$ iff $\mathbf{b} = I + \mathbf{S}(\mathbf{a})$ for $I \in \mathbb{Z}_n$.

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- $\mathbf{a} = \mathbf{b} - \mathbf{c}$ iff $\exists\ \sigma \in \mathbb{S}_k$ such that $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) - \sigma(\mathbf{c})$ where $\sigma(\mathbf{a}) = \sigma(\mathbf{a}_1) * \cdots * \sigma(\mathbf{a}_m)$ with $\sigma(\mathbf{a}_i)$ zero sum irred and $\sigma(\mathbf{b}) = \sigma(\mathbf{b}_1) * \cdots * \sigma(\mathbf{b}_m)$, $\sigma(\mathbf{c}) = \sigma(\mathbf{c}_1) * \cdots * \sigma(\mathbf{c}_m)$ where $\sigma(\mathbf{c}_i)$ 1-right shift of $\sigma(\mathbf{b}_i)$
- $\mathbf{a} = \mathbf{b} - \mathbf{c}$. Then $\mathbf{c}$ is 1-right shift of $\mathbf{b}$ iff $\mathbf{b} = l + \mathbf{S}(\mathbf{a})$ for $l \in \mathbb{Z}_n$.
- Then $\sigma(\mathbf{b}) = l_1 + \mathbf{S}(\mathbf{a}_1) * \cdots * l_m + \mathbf{S}(\mathbf{a}_m)$ for $l_1, \ldots, l_m \in \mathbb{Z}_n$

Putting Everything Together

Theorem

Let $\mathbf{a} = a_1, \ldots, a_k$ *be a sequence of elements in* $\mathbb{Z}_n$*. Then* $\mathbf{a} = \mathbf{b} - \mathbf{c}$ *for some sequences* $\mathbf{b} = b_1, \ldots, b_k$ *and* $\mathbf{c} = c_1, \ldots, c_k$ *that are rearrangements of each other if and only if* $\mathbf{a}$ *is a zero sum sequence and* $\mathbf{b} = \sigma^{-1}(l_1 + \mathbf{S}(\sigma(\mathbf{a}_1)) * l_2 + \mathbf{S}(\sigma((\mathbf{a}_2)) * \cdots * l_m + \mathbf{S}(\sigma(\mathbf{a}_m)))$ *where* $\sigma \in \mathbb{S}_k$*,* $l_1, \ldots, l_m \in \mathbb{Z}_n$ *and* $1 \leq m \leq k$*, and every* $\mathbf{c}_i$ *is a 1-right shift of* $\mathbf{b}_i$*.*

Putting Everything Together

Theorem

Let $\mathbf{a} = a_1, \ldots, a_k$ *be a sequence of elements in* $\mathbb{Z}_n$*. Then* $\mathbf{a} = \mathbf{b} - \mathbf{c}$ *for some sequences* $\mathbf{b} = b_1, \ldots, b_k$ *and* $\mathbf{c} = c_1, \ldots, c_k$ *that are rearrangements of each other if and only if* $\mathbf{a}$ *is a zero sum sequence and* $\mathbf{b} = \sigma^{-1}(l_1 + \mathbf{S}(\sigma(\mathbf{a}_1)) * l_2 + \mathbf{S}(\sigma((\mathbf{a}_2)) * \cdots * l_m + \mathbf{S}(\sigma(\mathbf{a}_m)))$ *where* $\sigma \in \mathbb{S}_k$*,* $l_1, \ldots, l_m \in \mathbb{Z}_n$ *and* $1 \leq m \leq k$*, and every* $\mathbf{c}_i$ *is a 1-right shift of* $\mathbf{b}_i$*.*

Corollary

Let $a : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ be a function, $\mathbf{a} = a_1, \ldots, a_n$ be a sequence such that $a(i) = a_i$ and M a multiset of elements in $\mathbb{Z}_n$. Then $a = b - c$ for function $b, c : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ such that $\{\{b(i)\}\} = \{\{c(i)\}\} = M$ if and only if $\mathbf{a}$ is zero sum and $M = \{\{l_1 + \mathbf{S}(\sigma(\mathbf{a}_1))\}\} \cup \cdots \cup \{\{l_m + \mathbf{S}(\sigma(\mathbf{a}_m)))\}\}$ for some $\sigma \in \mathbb{S}_n$, $l_1, \ldots, l_m \in \mathbb{Z}_n$

Future Directions and Problems to Solve

Open Problem

Given a zero sum sequence $\mathbf{a} = a_1, \ldots, a_n$ of elements in $\mathbb{Z}_n$ that can be written as $\mathbf{a} = \mathbf{a}_1 * \cdots * \mathbf{a}_m$ where each $\mathbf{a}_i$ is zero sum irreducible, what are the conditions on $\mathbf{a}$ that allow the existence of $l_1, \ldots, l_m \in \mathbb{Z}_n$ such that $\{l_1 + \mathbf{S}(\mathbf{a}_1) * \cdots * l_m + \mathbf{S}(\mathbf{a}_m)\} = \mathbb{Z}_n$?

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- Use result to construct cyclic group orthomorphisms
- Use result to construct transversals on the Cayley Table Latin Square

Enze WU Yifan ZHOU Yunpeng WANG Bin YU, Sijia REN.
Optimization of urban bus operation frequency under common route condition with rail transit.
Frontiers of Engineering Management, 4(4):451, 2017.

Paul Erdős, A. Ginzburg, and A. Ziv.
A theorem in additive number theory.
Bull. Res. Council Israel 10F, 41-43 (1961)., 1961.

Anthony B. Evans.
Applications of complete mappings and orthomorphisms of finite groups.
Quasigroups and Related Systems, 23(1):5–30, 2015.

Weidong Gao and Alfred Geroldinger.
Zero-sum problems in finite abelian groups: A survey.
Expositiones Mathematicae, 24(4):337 – 369, 2006.

Ron Graham.
Juggling Mathematics and Magic.

pages 1–6, 2013.

Marshal Hall.
A combinatorial problem in Abelian groups.
Proceedings of the American Mathematical Society, 3(4):584–587, 1952.

Douglas S. Stones and Ian M. Wanless.
Compound orthomorphisms of the cyclic group.
Finite Fields and their Applications, 16(4):277–289, 2010.

Daniel H. Ullman and Daniel J. Velleman.
Differences of bijections.
The American Mathematical Monthly, 126(3):199–216, 2019.

Acknowledgements

We would like to thank Dr. Gary Mullen for suggesting the problem.

An Application

Let $\mathbf{a} = 0, 1, 2, 3, 4, 5, 6,\ a_i \in \mathbb{Z}_7$

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$0, 1, 2, 3, 4, 5, 6 + 0, 1, 3, 5, 2, 6, 4 = 0, 2, 5, 1, 6, 4, 3$