

Permutation binomials of the form $x^r(x^{q-1} + a)$ over \mathbb{F}_{q^e}

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Abstract

We present several existence and nonexistence results for permutation binomials of the form $x^r(x^{q-1} + a)$, where $e \geq 2$ and $a \in \mathbb{F}_{q^e}^*$. As a consequence, we obtain a complete characterization of such permutation binomials over \mathbb{F}_{q^2} , \mathbb{F}_{q^3} , \mathbb{F}_{q^4} , \mathbb{F}_{p^5} , and \mathbb{F}_{p^6} , where p is an odd prime.

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1. Introduction

Let q be a prime power and \mathbb{F}_q denote the finite field of order q . Any function from \mathbb{F}_q to \mathbb{F}_q is uniquely represented by a polynomial of degree less than q via the modular reduction $x^q - x$. A polynomial in $\mathbb{F}_q[x]$ is a *permutation polynomial* over \mathbb{F}_q if it induces a bijection in \mathbb{F}_q .

The study of permutation polynomials over finite fields dates back to the 1800s with the work of Hermite [1]. Since then, many researchers have investigated them, not only because they are interesting on their own, but also due to their applications in areas such as coding theory, cryptography, and finite geometry. Permutation monomials have been completely characterized as x^r permutes \mathbb{F}_q if and only if $\gcd(r, q-1) = 1$. In the absence of a general and simple permutation criterion for binomials, permutation binomials have been the focus of many studies. As a matter of fact, the majority of the results on permutation polynomials concern permutation binomials as they already impose many challenges, despite their simple form. Given $x^m + ax^n$ in $\mathbb{F}_q[x]$, it is desirable to find simple relationships between m , n , a , and q that make the binomial permute \mathbb{F}_q . This is usually

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achieved by specializing these parameters to particular forms. We refer the interested readers to the surveys [2, 3, 4] for an excellent overview of the area.

Besides the number of terms of the polynomial, there are other parameters such as the degree, the Carlitz rank, and the index that are relevant in many problems involving polynomials over finite fields. The concept of *index* was introduced by Akbary, Ghioca, and Wang [5] in 2009; we postpone the precise definition to Section 2. It turns out that any nonconstant polynomial of index ℓ and degree at most $q - 1$ in $\mathbb{F}_q[x]$ can be uniquely expressed as $\alpha x^r h(x^{(q-1)/\ell}) + \beta$. Wang [6] wrote a survey on permutation polynomials classified by their indices, indicating that this might be a promising way of organizing the many existing results. For instance, a result by Masuda and Zieve in [7] can be stated as: the index of any permutation binomial over the prime field \mathbb{F}_p is always less than $\sqrt{p} - 1$.

Wang also suggested a list of open problems in his survey. Problem 7 is “*Construct and classify permutation polynomials of \mathbb{F}_{q^e} with intermediate indices such as $q^{e-1} + \dots + q + 1$, $c(q^{e-1} + \dots + q + 1)$, or $(q^{e-1} + \dots + q + 1)/d$, where c is a positive factor of $q - 1$, and d is a positive factor of $q^{e-1} + \dots + q + 1$.*” Not only there are not many results in this direction, the ones listed in [6] for $e \geq 3$ involve polynomials with more than two terms.

We now turn our attention to permutation binomials of a special type. Let S be the collection of all $(q^e, r, t, a) \in \mathbb{N}^3 \times \mathbb{F}_{q^e}^*$, $e \geq 2$, such that $x^r(x^{t(q-1)} + a)$ permutes \mathbb{F}_{q^e} . Most of the results in the literature deal with $e = 2$ for which the index is $(q + 1)/\gcd(q + 1, t)$ from Problem 7. In this case, under the assumptions that $\gcd(rp, t(q - 1)) = 1$, where $p = \text{char } \mathbb{F}_q$ and $(-a)^{(q+1)/\gcd(q+1,t)} \neq 1$, Hou [8] grouped several results that appear in [2, 9, 10, 11, 12, 13] into the following four families:

- (i) (q^2, r, t, a) , $a^{q+1} = 1$, $\gcd(r - t, q + 1) = 1$
- (ii) $(q^2, r, 1, a)$, $r \equiv 1 \pmod{q + 1}$
- (iii) $(q^2, 1, 2, a)$, $(-a)^{(q+1)/2} = 3$
- (iv) $(q^2, 3, 2, a)$, $(-a)^{(q+1)/2} = 1/3$

and conjectured that these are the only infinite ones in S with $e = 2$.

In this paper we investigate conditions for $(q^e, r, 1, a)$ to belong to S . In other words, we consider binomials of the form $f(x) = x^r(x^{q-1} + a)$ over \mathbb{F}_{q^e} for $e \geq 2$. When $r \leq q^e - q$, the index of $f(x)$ is $\ell = q^{e-1} + \dots + q + 1$, so our results contribute to the solution of Problem 7. An immediate necessary condition is that $(-a)^\ell \neq 1$, since otherwise $f(x)$ would have more than one root. Hou’s

second family concerns the case $e = 2$, and Liu studied the case $e = 3$. We state their results as follows.

Proposition 1.1. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \in \{2, 3\}$ and $a \neq 0$, and let $\ell = q^{e-1} + \cdots + q + 1$.*

- (i) [8, Theorem 4.2] *For $r \geq 1$, $f(x)$ permutes \mathbb{F}_{q^2} if and only if $a^\ell \neq 1$, $\gcd(r, q-1) = 1$, and $r \equiv 1 \pmod{\ell}$.*
- (ii) [14, Theorem 1] *For q odd and $1 \leq r \leq \ell$, $f(x)$ permutes \mathbb{F}_{q^3} if and only if $(-a)^\ell \neq 1$ and $r = 1$.*

Li et al. [15, Theorem 3.1] showed a weaker version of Proposition 1.1(i) by considering $1 \leq r \leq \ell$, which is the same restriction that appears in (ii). In both cases, the authors restricted r to such values due to the following result.

Proposition 1.2. [15, Lemma 2.2] *Let $f(x) = x^r(x^{(q^e-1)/\ell} + a) \in \mathbb{F}_{q^e}[x]$ with $a \neq 0$ and $\ell \mid q^e - 1$. If $f(x)$ permutes \mathbb{F}_{q^e} and $\gcd(r + \ell, (q^e - 1)/\ell) = 1$, then $x^{r+\ell}(x^{(q^e-1)/\ell} + a)$ also permutes \mathbb{F}_{q^e} .*

We observe that Proposition 1.1(ii) together with Proposition 1.2 may not produce all permutation binomials $x^r(x^{q-1} + a)$ over \mathbb{F}_{q^3} . For instance, the permutation binomials of degree less than 26 over \mathbb{F}_{27} are $x(x^2 + a)$ and $x^{23}(x^2 + a)$, with $(-a)^{13} \neq 1$. While the former one is the only permutation binomial given by Proposition 1.1(ii), the latter one is not given by Proposition 1.2. In fact, Proposition 1.2 cannot be applied to $x(x^2 + a)$ since $\gcd(r + \ell, (q^e - 1)/\ell) = \gcd(14, 2) = 2$. Moreover, we observe that, in order to obtain $x^{23}(x^2 + a)$ from Proposition 1.2, we would need $x^{10}(x^2 + a)$ to permute \mathbb{F}_{27} , which is clearly not true. In particular, this shows that the converse of Proposition 1.2 does not hold. One of our goals in this paper is to extend Proposition 1.1(ii) to include the values of r such that $r > \ell$.

We show several existence and nonexistence results of permutation binomials of the form $x^r(x^{q-1} + a)$ over \mathbb{F}_{q^e} for $e \geq 2$. As a consequence, we completely characterize such permutation binomials over $\mathbb{F}_{q^2}, \mathbb{F}_{q^3}, \mathbb{F}_{q^4}, \mathbb{F}_{p^5}$, and \mathbb{F}_{p^6} , where p is an odd prime. In particular, we extend Proposition 1.1. Even though the case \mathbb{F}_{q^2} has been resolved by Hou, we include it in our characterization, since we state our result in terms of ℓ , which allows us to have a statement similar to Proposition 1.1(i) that holds for other values of e . Below we summarize the results. We denote the (nonnegative) reduction of r modulo ℓ by $r \pmod{\ell}$.

Theorem 1.3. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $2 \leq e \leq 6$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$.*

- (i) *When $e = 2, 3, 4$, $f(x)$ permutes \mathbb{F}_{q^e} if and only if $(-a)^\ell \neq 1$, $\gcd(r, q-1) = 1$, and $r \pmod{\ell} \in \{1, \ell - q\}$.*
- (ii) *When $e = 5$ and q is an odd prime, $f(x)$ permutes \mathbb{F}_{q^5} if and only if $(-a)^\ell \neq 1$, $\gcd(r, q-1) = 1$, and $r \pmod{\ell} \in \{1, \ell - q, q^3 + 1, q^4 + q^2 + 1\}$.*
- (iii) *When $e = 6$ and q is an odd prime, $f(x)$ permutes \mathbb{F}_{q^6} if and only if $(-a)^\ell \neq 1$, $\gcd(r, q-1) = 1$, and $r \pmod{\ell} \in \{1, \ell - q\}$.*

In the next result we construct permutation binomials $f(x)$ for arbitrary e .

Theorem 1.4. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. Then $f(x)$ permutes \mathbb{F}_{q^e} and is the composition of a linearized binomial and a monomial if and only if $(-a)^\ell \neq 1$ and $r = s\ell + \sum_{i=0}^{k-1} q^{hi} \pmod{q^e - 1}$, where $\gcd(h, e) = 1$, $k \pmod{e} = h^{-1}$, s is a positive integer, and $\gcd(r, q-1) = 1$.*

Theorem 1.4 shows that, for any q^e with $e \geq 2$, there is always a 4-tuple $(q^e, r, 1, a)$ in S . So this yields an infinite family in S that extends Hou's second family described in Proposition 1.1(i). Moreover, we have run a computer search for $q^e < 10^8$ that leads us to conjecture that our family is the only one in S for which $t = 1$.

Conjecture 1.5. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. Then $f(x)$ permutes \mathbb{F}_{q^e} if and only if $f(x)$ is congruent to the composition of a linearized binomial $L(x) = x^{q^h} + ax$ and the monomial x^r modulo $x^{q^e} - x$, where $(-a)^\ell \neq 1$ and $\gcd(r, q-1) = 1$.*

Theorems 1.3 and 1.4 show that this conjecture is true when $e \in \{2, 3, 4\}$ for any q , and when $e \in \{5, 6\}$ for any odd prime q . One direction of Theorem 1.3 is obtained through a series of nonexistence results. In addition, these results collectively restrict the possible values of r for the remaining values of q^e .

We organize this paper as follows. In Section 2 we present some tools that we use in Section 4, including a permutation criterion and some technical results involving binomial coefficients. The goal of Section 3 is to show Theorem 1.4. In Section 4 we present several nonexistence results. Finally, in Section 5 we apply some results from Sections 3 and 4 to show Theorem 1.3 as a combination of Propositions 5.1 to 5.3. We end the paper with a discussion on Conjecture 1.5.

2. Preliminaries

We start this section by defining the index of a polynomial $f(x) \in \mathbb{F}_{q^e}[x]$ of degree at most $q^e - 1$. Write

$$f(x) = \alpha(x^d + a_{d-i_1}x^{d-i_1} + \cdots + a_{d-i_k}x^{d-i_k}) + \beta,$$

where $\alpha, a_{d-i_j} \neq 0$ for $j = 1, \dots, k$. We suppose that $k \geq 1$ and $d - i_k = r$. The integer

$$\ell = \frac{q^e - 1}{\gcd(d - r, d - r - i_1, \dots, d - r - i_{k-1}, q^e - 1)}$$

is called the *index* of $f(x)$. We note that when $r \leq q^e - q$ the index of $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ is $\ell = q^{e-1} + \cdots + q + 1$.

Our results rely on a variation of Hermite's criterion obtained by Masuda, Panario, and Wang, namely Theorem 1 in [16]. The authors specialized their criterion for polynomials to binomials. The result is immediate, and it appears as Corollary 2 without a proof. In both cases, they require the degree of the polynomial in $\mathbb{F}_{q^e}[x]$ to be less than $q^e - 1$. We examined the proof of Theorem 1, and checked that this condition is not needed. Below we state their criterion for permutation binomials without the degree requirement.

Proposition 2.1. [16, Corollary 2] *Let $f(x) = x^r(x^s + a) \in \mathbb{F}_{q^e}[x]$ with $q^e \geq 3$, $a \neq 0$, and $r, s \geq 1$. Then $f(x)$ permutes \mathbb{F}_{q^e} if and only if*

$$\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \begin{cases} 0 & \text{if } N = 1, \dots, q^e - 2 \\ 1 & \text{if } N = q^e - 1, \end{cases} \quad (1)$$

where

$$S_N = \left\{ A \in \mathbb{Z} : A = \frac{j(q^e - 1) - rN}{s}, \text{ where } j \in \mathbb{Z} \text{ and } 0 \leq A \leq N \right\}.$$

For convenience, we may write $S_{N,r}$, A_j or $A_{j,r}$ to emphasize the parameters involved. For binomials $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ and $\ell = q^{e-1} + \cdots + q + 1$, we have

$$S_N = \left\{ A_j \in \mathbb{Z} : A_j = j\ell - \frac{rN}{q-1}, \text{ where } j \in \mathbb{Z} \text{ and } 0 \leq A_j \leq N \right\}. \quad (2)$$

The next lemma provides the only two possible values for $|S_N|$ when $rN/(q-1)$ is an integer. The proof gives the exact conditions for each value to occur. Since we do not use these conditions, for simplicity, we opt to state the result without them.

Lemma 2.2. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. If $q - 1 \mid rN$ for some $N \in \{1, \dots, q^e - 1\}$, then $|S_N| = \lfloor N/\ell \rfloor$ or $\lfloor N/\ell \rfloor + 1$.*

Proof. By (2), we have

$$\begin{aligned} |S_N| &= \left| \left\{ j \in \mathbb{Z}: 0 \leq j\ell - \frac{rN}{q-1} \leq N \right\} \right| \\ &= \left| \left\{ j \in \mathbb{Z}: \left\lceil \frac{rN}{\ell(q-1)} \right\rceil \leq j \leq \left\lfloor \frac{rN}{\ell(q-1)} + \frac{N}{\ell} \right\rfloor \right\} \right|. \end{aligned}$$

The result follows from the fact that for $u, v \geq 0$ we have

$$|[[u], [u+v]] \cap \mathbb{Z}| = \begin{cases} |[u, u+[v]] \cap \mathbb{Z}| = [v] + 1 & \text{if } u \in \mathbb{Z} \\ |[u] + 1, [u] + [v]] \cap \mathbb{Z}| = [v] & \text{if } u \notin \mathbb{Z}, u - [u] + v - [v] < 1 \\ |[u] + 1, [u] + [v] + 1] \cap \mathbb{Z}| = [v] + 1 & \text{if } u \notin \mathbb{Z}, u - [u] + v - [v] \geq 1. \end{cases}$$

□

Following the idea underlying Proposition 1.2 with regard to reducing the possible values of r , we show that it suffices to consider $r \in \{1, \dots, \ell\}$ to show the nonexistence of permutation binomials in a special circumstance. This result will play an important role in our proofs in Section 4.

Lemma 2.3. *Let $e \geq 2$, $a \in \mathbb{F}_{q^e}^*$, and $\ell = q^{e-1} + \dots + q + 1$. Let r and s be positive integers such that $r \equiv s \pmod{\ell}$. Suppose that $x^s(x^{q-1} + a)$ does not permute \mathbb{F}_{q^e} . If $\sum_{A \in S_{N,s}} \binom{N}{A} a^{N-A} \neq 0$ for some $N \in \{1, \dots, q^e - 2\}$ such that $q - 1 \mid N$, then $x^r(x^{q-1} + a)$ does not permute \mathbb{F}_{q^e} .*

Proof. Let $A_{j,r} \in S_{N,r}$. Write $r = s + k\ell$, where k is an integer, and set $j' = j - kN/(q-1)$. Then

$$A_{j,r} = j\ell - r \frac{N}{q-1} = \left(j' + k \frac{N}{q-1} \right) \ell - (s + k\ell) \frac{N}{q-1} = j'\ell - s \frac{N}{q-1} = A_{j',s} \in S_{N,s},$$

so $S_{N,r} \subseteq S_{N,s}$. This also shows that $S_{N,s} \subseteq S_{N,r}$. Therefore $S_{N,r} = S_{N,s}$. By Proposition 2.1, we conclude that $x^r(x^{q-1} + a)$ does not permute \mathbb{F}_{q^e} . □

The next two results concern identities with binomial coefficients. Our convention is that $\binom{n}{k} = 0$ if $n < k$.

Proposition 2.4 (Lucas' Theorem). *Let $n \geq k$ be positive integers and p be a prime. If $n = n_m p^m + n_{m-1} p^{m-1} + \dots + n_0$ and $k = k_m p^m + k_{m-1} p^{m-1} + \dots + k_0$ are the p -adic expansions of n and k , then*

$$\binom{n}{k} \equiv \binom{n_m}{k_m} \cdots \binom{n_0}{k_0} \pmod{p}. \quad (3)$$

When q is a power of p , the identity (3) also holds with the q -adic expansions of n and k , namely, $n = n_m q^m + n_{m-1} q^{m-1} + \cdots + n_0$ and $k = k_m q^m + k_{m-1} q^{m-1} + \cdots + k_0$.

Lemma 2.5. *Let $q = p^m$, where p is prime, and let k be an integer such that $0 \leq k \leq q - 1$. Then*

- (i) $\binom{q-1}{k} \not\equiv 0 \pmod{p}$,
- (ii) $\binom{q-2}{k} \equiv 0 \pmod{p}$ if and only if $k \equiv p - 1 \pmod{p}$.

Proof. Let $k = k_{m-1} p^{m-1} + \cdots + k_0$ be the p -adic expansion of k .

- (i) Write $q - 1 = (p - 1)p^{m-1} + \cdots + (p - 1)p + (p - 1)$. By Lucas' Theorem, $\binom{q-1}{k} \equiv \binom{p-1}{k_{m-1}} \cdots \binom{p-1}{k_0} \not\equiv 0 \pmod{p}$, since $0 \leq k_i < p$.
- (ii) Write $q - 2 = (p - 1)p^{m-1} + \cdots + (p - 1)p + (p - 2)$. By Lucas' Theorem, $\binom{q-2}{k} \equiv \binom{p-1}{k_{m-1}} \cdots \binom{p-1}{k_1} \binom{p-2}{k_0} \pmod{p}$. Since $\binom{p-1}{k_i} \not\equiv 0 \pmod{p}$ for each i , we have $\binom{q-2}{k} \equiv 0 \pmod{p}$ if and only if $\binom{p-2}{k_0} \equiv 0 \pmod{p}$, which holds if and only if $k_0 = p - 1$.

□

3. Existence results

In this section we construct permutation binomials of the form $f(x) = x^r(x^{q-1} + a)$ over \mathbb{F}_{q^e} . Two immediate necessary conditions are that $\gcd(r, q - 1) = 1$ and $(-a)^\ell \neq 1$, where $\ell = q^{e-1} + \cdots + q + 1$. The former condition prevents us from writing $f(x)$ as a composition of a binomial and a permutation monomial x^n , where $n = \gcd(r, q - 1)$. Next we consider a more general version of the latter condition, and show that it is equivalent to having a family of linearized binomials that permute \mathbb{F}_{q^e} . The result is not new, but we include a proof for completeness.

Lemma 3.1. *Let $e \geq 2$ and $a \in \mathbb{F}_{q^e}^*$. Then $L(x) = x^{q^h} + ax$ permutes \mathbb{F}_{q^e} if and only if $(-a)^{(q^e-1)/(q^d-1)} \neq 1$, where $d = \gcd(e, h)$.*

Proof. The linearized binomial $L(x)$ permutes \mathbb{F}_{q^e} if and only if zero is its only root in \mathbb{F}_{q^e} . Let ξ be a primitive element of \mathbb{F}_{q^e} . Suppose that $L(\xi^k) = 0$ for some integer k , so $-a = \xi^{k(q^h-1)}$. Then $(-a)^{(q^e-1)/(q^d-1)} = (\xi^{q^e-1})^{k(q^h-1)/(q^d-1)} = 1$. Conversely, if $(-a)^{(q^e-1)/(q^d-1)} = 1$, then $-a = \xi^{r(q^d-1)}$ for some integer r . Since $\gcd(q^e-1, q^h-1) = q^d-1$, we write $q^d-1 = b(q^e-1) + c(q^h-1)$ for some integers b and c . Then $L(\xi^{rc}) = \xi^{rc} \left(\xi^{rc(q^h-1)} + a \right) = \xi^{rc} \left(\xi^{r(q^d-1)-rb(q^e-1)} + a \right) = 0$. □

Next we show that any binomial $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ that is congruent to the composition of a linearized binomial and a monomial modulo $x^{q^e} - x$ satisfies $f(x) \equiv L(x^r) \pmod{x^{q^e} - x}$.

Lemma 3.2. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$ and $a \neq 0$. Then $f(x) \equiv \bar{L}(x^s) \pmod{x^{q^e} - x}$, where $\bar{L}(x) = bx^{q^m} + cx^{q^n}$ for positive integers $s, m,$ and n with $n < m < e$ if and only if $f(x) \equiv L(x^r) \pmod{x^{q^e} - x}$, where $L(x) = x^{q^h} + ax$ for some positive integer h .*

Proof. We have $f(x) \equiv \bar{L}(x^s) \pmod{x^{q^e} - x}$, that is, $x^{r+q-1} + ax^r \equiv bx^{sq^m} + cx^{sq^n} \pmod{x^{q^e} - x}$, which holds in one of the following situations:

- (1) $b = 1, a = c, r + q - 1 \equiv sq^m \pmod{q^e - 1}$, and $r \equiv sq^n \pmod{q^e - 1}$, or
- (2) $c = 1, a = b, r + q - 1 \equiv sq^n \pmod{q^e - 1}$, and $r \equiv sq^m \pmod{q^e - 1}$.

We want to show that there exists a positive integer h such that $rq^h \equiv r + q - 1 \pmod{q^e - 1}$. It suffices to choose $h = m - n$ in Case (1), and $h = e + n - m$ in Case (2). This concludes the proof as the converse follows immediately. \square

In order to find all binomials $f(x) = x^r(x^{q-1} + a)$ that are congruent to compositions of $L(x) = x^{q^h} + ax$ and x^r modulo $x^{q^e} - x$, we need to solve $r(q^h - 1) - q + 1 \equiv 0 \pmod{q^e - 1}$ for h and r .

Lemma 3.3. *Let $e \geq 2$ and $\ell = q^{e-1} + \dots + q + 1$. Then $r(q^h - 1) - q + 1 \equiv 0 \pmod{q^e - 1}$ if and only if $\gcd(h, e) = 1$ and $r \equiv s\ell + \sum_{i=0}^{k-1} q^{hi} \pmod{q^e - 1}$, where $k \pmod{e} = h^{-1}$ and s is any integer.*

Proof. We first observe that $r \equiv s\ell + \sum_{i=0}^{k-1} q^{hi} \pmod{q^e - 1}$ satisfies

$$\begin{aligned} r(q^h - 1) - q + 1 &\equiv s\ell(q^h - 1) + \sum_{i=1}^k q^{hi} - \sum_{i=0}^{k-1} q^{hi} - q + 1 \\ &\equiv q^{hk} - q \equiv 0 \pmod{q^e - 1}. \end{aligned}$$

Conversely, suppose that $r(q^h - 1) - q + 1 \equiv 0 \pmod{q^e - 1}$. So $r \left(\frac{q^h - 1}{q - 1} \right) \equiv 1 \pmod{\frac{q^e - 1}{q - 1}}$ implies that $\gcd\left(\frac{q^h - 1}{q - 1}, \frac{q^e - 1}{q - 1}\right) = 1$, that is, $\gcd(h, e) = 1$. Let $\bar{r} = \sum_{i=0}^{k-1} q^{hi} \pmod{q^e - 1}$. Then, similarly to the calculations above, $\bar{r}(q^h - 1) - q + 1 \equiv 0 \pmod{q^e - 1}$, so \bar{r} is also an inverse of $\frac{q^h - 1}{q - 1}$ modulo ℓ . Thus $r \equiv \bar{r} \pmod{\ell}$, i.e., $r \equiv s\ell + \sum_{i=0}^{k-1} q^{hi} \pmod{q^e - 1}$ for any integer s . \square

Theorem 3.4 is our main tool to construct permutation binomials of the form $x^r(x^{q-1} + a)$ over \mathbb{F}_{q^e} . We find all r 's that give permutation binomials that are compositions of a linearized binomial and a monomial. For these values of r , we require r to be coprime to $q - 1$ as we can show that this implies that r is coprime to $q^e - 1$.

Theorem 3.4. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. Then $f(x)$ permutes \mathbb{F}_{q^e} and is the composition of a linearized binomial and a monomial if and only if $(-a)^\ell \neq 1$ and $r = s\ell + \sum_{i=0}^{k-1} q^{hi} \pmod{q^e - 1}$, where $\gcd(h, e) = 1$, $k \pmod{e} = h^{-1}$, s is a positive integer, and $\gcd(r, q - 1) = 1$.*

Proof. The combination of the three previous lemmas implies that we only need to show that $\gcd(r, q^e - 1) = 1$ to complete the ‘if’ direction. By Lemma 3.3, we have $r(q^h - 1) - q + 1 \equiv 0 \pmod{q^e - 1}$. By writing $r(q^h - 1) - q + 1 = (q^e - 1)t$ for an integer t and dividing both sides by $q - 1$, we obtain that $r \sum_{i=0}^{h-1} q^i - t\ell = 1$, so $\gcd(r, \ell) = 1$, which means that $\gcd(r, q^e - 1) = 1$. \square

Table 1 displays the values of r given by Theorem 3.4, for $2 \leq e \leq 8$. We also list their corresponding reductions modulo ℓ to illustrate that they are the same values that appear in Theorem 1.3, for $2 \leq e \leq 6$, showing one direction of our characterization. This direction can be also obtained from the next result that is more general. The other direction is proved in Section 5.

Corollary 3.5. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. Suppose that $(-a)^\ell \neq 1$ and $\gcd(r, q - 1) = 1$.*

- (i) *If $r \pmod{\ell} \in \{1, \ell - q\}$, then $f(x)$ permutes \mathbb{F}_{q^e} .*
- (ii) *If e is odd and $r \pmod{\ell} \in \{q^{(e+1)/2} + 1, \ell - q^{(e+1)/2} - q\}$, then $f(x)$ permutes \mathbb{F}_{q^e} .*

Proof. (i) Apply Theorem 3.4 with $h = 1$ and $h = e - 1$

(ii) Apply Theorem 3.4 with $h = (e + 1)/2$ and $h = (e - 1)/2$.

\square

When $e = 3$, the two sets provided by Corollary 3.5 are the same, i.e., $\{1, \ell - q\} = \{q^{(e+1)/2} + 1, \ell - q^{(e+1)/2} - q\}$.

e	ℓ	h	k	r	$r \pmod{\ell}$
2	$q+1$	1	1	$s\ell+1$	1
3	q^2+q+1	1	1	$s\ell+1$	1
		2	2	$s\ell+q^2+1$	$\ell-q$
4	q^3+q^2+q+1	1	1	$s\ell+1$	1
		3	3	$s\ell+q^6+q^3+1$ $\equiv s\ell+q^3+q^2+1 \pmod{q^4-1}$	$\ell-q$
5	$q^4+q^3+q^2+q+1$	1	1	$s\ell+1$	1
		2	3	$s\ell+q^4+q^2+1$	q^4+q^2+1
		3	2	$s\ell+q^3+1$	q^3+1
		4	4	$s\ell+q^{12}+q^8+q^4+1$ $\equiv s\ell+q^4+q^3+q^2+1 \pmod{q^5-1}$	$\ell-q$
6	$q^5+q^4+q^3+q^2+q+1$	1	1	$s\ell+1$	1
		5	5	$s\ell+q^{20}+q^{15}+q^{10}+q^5+1$ $\equiv s\ell+q^5+q^4+q^3+q^2+1 \pmod{q^6-1}$	$\ell-q$
7	$q^6+q^5+q^4+q^3+q^2+q+1$	1	1	$s\ell+1$	1
		2	4	$s\ell+q^6+q^4+q^2+1$	$q^6+q^4+q^2+1$
		3	5	$s\ell+q^{12}+q^9+q^6+q^3+1$ $\equiv s\ell+q^6+q^5+q^3+q^2+1 \pmod{q^7-1}$	$\ell-q^4-q$
		4	2	$s\ell+q^4+1$	q^4+1
		5	3	$s\ell+q^{10}+q^5+1$ $\equiv s\ell+q^5+q^3+1 \pmod{q^7-1}$	q^5+q^3+1
		6	6	$s\ell+q^{30}+q^{24}+q^{18}+q^{12}+q^6+1$ $\equiv s\ell+q^6+q^5+q^4+q^3+q^2+1 \pmod{q^7-1}$	$\ell-q$
8	$q^7+q^6+q^5+q^4+q^3+q^2+q+1$	1	1	$s\ell+1$	1
		3	3	$s\ell+q^6+q^3+1$	q^6+q^3+1
		5	5	$s\ell+q^{20}+q^{15}+q^{10}+q^5+1$ $\equiv s\ell+q^7+q^5+q^4+q^2+1 \pmod{q^8-1}$	$\ell-q^6-q^3-q$
		7	7	$s\ell+q^{42}+q^{35}+q^{28}+q^{21}+q^{14}+q^7+1$ $\equiv s\ell+q^7+q^6+q^5+q^4+q^3+q^2+1 \pmod{q^8-1}$	$\ell-q$

Table 1: Values of r given by Theorem 3.4 and their reductions modulo ℓ , for $2 \leq e \leq 8$.

4. Nonexistence results

In this section we use Proposition 2.1 to show the nonexistence of permutation binomials of the form $f(x) = x^r(x^{q-1} + a)$ over \mathbb{F}_{q^e} for $e \geq 2$ and $q \neq 2$. When $q = 2$, the binomial takes the form $f(x) = x^r(x+a)$, so it does not permute \mathbb{F}_{2^e} if $a \neq 0$. To calculate the sum (1) with A_j running over S_N (2), we apply Lucas' Theorem using q -adic expansions. Our first nonexistence result narrows down the possibilities for $f(x)$ to those with $r \pmod{\ell} = hq + 1$ for some integer h .

Proposition 4.1. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. If $r \pmod{\ell} \neq hq + 1$ for any integer h , then $f(x)$ does not permute \mathbb{F}_{q^e} .*

Proof. First, assume that $1 \leq r < \ell$ and $r \neq hq + 1$ for $0 \leq h < \sum_{i=0}^{e-2} q^i$. Then $r \neq 1, q + 1, 2q + 1, \dots, \left(\sum_{i=0}^{e-2} q^i\right)q + 1$. In other words, $h_0q + 2 \leq r \leq (h_0 + 1)q$ for some $0 \leq h_0 \leq \sum_{i=1}^{e-2} q^i$. We claim that $S_N = \{A_{h_0+1}\}$ for $N = q^{e-1} - 1$. By Lemma 2.2, since $N/\ell = (q^{e-1} - 1)/(\sum_{i=0}^{e-1} q^i) < 1$, it follows that $|S_N| \leq 1$. We have

$$A_{h_0+1} = (h_0 + 1) \sum_{i=0}^{e-1} q^i - r \sum_{i=0}^{e-2} q^i.$$

Since $r \leq (h_0 + 1)q$,

$$A_{h_0+1} \geq (h_0 + 1) \sum_{i=0}^{e-1} q^i - (h_0 + 1) \sum_{i=1}^{e-1} q^i = h_0 + 1 > 0.$$

Also, the conditions $r \geq h_0q + 2$ and $h_0 \leq \sum_{i=1}^{e-2} q^i$ imply that

$$A_{h_0+1} \leq (h_0 + 1) \sum_{i=0}^{e-1} q^i - (h_0q + 2) \sum_{i=0}^{e-2} q^i = h_0 + q^{e-1} - \sum_{i=0}^{e-2} q^i \leq q^{e-1} - 1 = N.$$

Hence $S_N = \{A_{h_0+1}\}$. Write $N = (q - 1)q^{e-2} + \dots + (q - 1)q + q - 1$. By Lucas' Theorem and Lemma 2.5(i), we obtain that

$$\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \binom{N}{A_{h_0+1}} a^{N-A_{h_0+1}} \neq 0.$$

Therefore $f(x)$ does not permute \mathbb{F}_{q^e} for $1 \leq r < \ell$.

For $r = \ell$, let $j = N/(q - 1)$. Then $A_j = 0$, $S_N = \{0\}$, $\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \binom{N}{0} a^N \neq 0$, and $f(x)$ does not permute \mathbb{F}_{q^e} . Since $q - 1 \mid N$, by Lemma 2.3 $f(x)$ does not permute \mathbb{F}_{q^e} when $r > \ell$. This completes the proof. \square

The next two propositions give more precise descriptions of $r \pmod{\ell} = hq + 1$ for which $f(x) = x^r(x^{q-1} + a)$ is a permutation binomial by providing conditions on the divisibility of h . We begin by showing that for e even we must have $q + 1 \mid h$.

Proposition 4.2. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with e even and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. If $r \pmod{\ell} = hq + 1$ with $q + 1 \nmid h$, then $f(x)$ does not permute \mathbb{F}_{q^e} .*

Proof. We assume that $1 \leq r < \ell$ and $r = hq + 1$, where $h = k(q + 1) + h_0$ with positive integers k and h_0 , and $1 \leq h_0 \leq q$. Let $N = (q - 1) \sum_{i=0}^{(e-2)/2} q^{2i}$. Since $N < \ell$, Lemma 2.2 implies that S_N has at most one element. We compute A_{kq+h_0}

$$\begin{aligned}
&= (kq + h_0) \sum_{i=0}^{e-1} q^i - [(k(q + 1) + h_0)q + 1] \sum_{i=0}^{(e-2)/2} q^{2i} \\
&= kq \sum_{i=0}^{e-1} q^i + h_0 \sum_{i=0}^{e-1} q^i - kq^2 \sum_{i=0}^{(e-2)/2} q^{2i} - kq \sum_{i=0}^{(e-2)/2} q^{2i} \\
&\quad - h_0q \sum_{i=0}^{(e-2)/2} q^{2i} - \sum_{i=0}^{(e-2)/2} q^{2i} \\
&= k \left(\sum_{i=1}^e q^i - \sum_{i=1}^{e/2} q^{2i} - \sum_{i=0}^{(e-2)/2} q^{2i+1} \right) + h_0 \left(\sum_{i=0}^{e-1} q^i - \sum_{i=0}^{(e-2)/2} q^{2i+1} \right) - \sum_{i=0}^{(e-2)/2} q^{2i} \\
&= (h_0 - 1) \sum_{i=0}^{(e-2)/2} q^{2i},
\end{aligned}$$

so $0 \leq A_{h_0q+h_0} \leq N$, and thus $S_N = \{A_{kq+h_0}\}$. By Lucas' Theorem and Lemma 2.5(i), we conclude that

$$\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \binom{N}{A_{kq+h_0}} a^{N-A_{kq+h_0}} \neq 0.$$

Therefore $f(x)$ does not permute \mathbb{F}_{q^e} . Since $q - 1 \mid N$, we use Lemma 2.3 to obtain the desired result when $r \geq \ell$. \square

We now show that if q is a power of an odd prime p and $f(x)$ permutes \mathbb{F}_{q^e} , then $p \mid h$.

Proposition 4.3. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$, q a power of an odd prime p , and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. If $r \pmod{\ell} = hq + 1$ and $p \nmid h$, then $f(x)$ does not permute \mathbb{F}_{q^e} .*

Proof. We start by assuming that $r < \ell$ and $r = hq + 1$, where $0 \leq h < \sum_{i=0}^{e-2} q^i$ and $p \nmid h$. Let $N = 2q^{e-1} - 2$. By Lemma 2.2, since $N/\ell = 2(q^{e-1} - 1)/(\sum_{i=0}^{e-1} q^i) < 2$, we have $|S_N| \leq 2$. We claim that A_{2h+1} is the only element in S_N . To show this, we compute

$$A_{2h+1} = (2h+1) \sum_{i=0}^{e-1} q^i - 2(hq+1) \sum_{i=0}^{e-2} q^i = q^{e-1} - \sum_{i=0}^{e-2} q^i + 2h.$$

Since $h \geq 0$, we obtain that

$$A_{2h+1} \geq q^{e-1} - \sum_{i=0}^{e-2} q^i = q^{e-1} - \frac{q^{e-1} - 1}{q - 1} > 0.$$

On the other hand, the condition $h \leq \sum_{i=0}^{e-2} q^i - 1$ implies that

$$A_{2h+1} \leq q^{e-1} - \sum_{i=0}^{e-2} q^i + 2 \left(\sum_{i=0}^{e-2} q^i - 1 \right) = q^{e-1} + \sum_{i=0}^{e-2} q^i - 2 < q^{e-1} + q^{e-1} - 2 = N.$$

Hence $A_{2h+1} \in S_N$. We observe that $A_{j+k} = A_j + k\ell$ for any integers j and k , so it remains to show that A_{2h} and A_{2h+2} are not in $[0, N]$. In fact, $A_{2h} = A_{2h+1} - \ell$

$$= q^{e-1} - \sum_{i=0}^{e-2} q^i + 2h - \sum_{i=0}^{e-1} q^i < q^{e-1} - \sum_{i=0}^{e-2} q^i + 2 \left(\sum_{i=0}^{e-2} q^i - 1 \right) - \sum_{i=0}^{e-1} q^i = -2$$

and

$$A_{2h+2} = A_{2h+1} + \ell = q^{e-1} - \sum_{i=0}^{e-2} q^i + 2h + \sum_{i=0}^{e-1} q^i = 2q^{e-1} + 2h > N.$$

Therefore $S_N = \{A_{2h+1}\}$.

Write $N = q^{e-1} + (q-1)q^{e-2} + \dots + (q-1)q + (q-2)$ and $A_{2h+1} = a_{e-1}q^{e-1} + \dots + a_1q + a_0$ in base q . Since p is an odd prime and $p \nmid h$, it follows that $A_{2h+1} \equiv 2h - 1 \not\equiv p - 1 \pmod{p}$ so that $\binom{q-2}{a_0} \not\equiv 0 \pmod{p}$. By Lucas' Theorem and Lemma 2.5, we get

$$\begin{aligned} \sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_{2h+1}} a^{N-A_{2h+1}} \\ &= \binom{1}{a_{e-1}} \binom{q-1}{a_{e-2}} \dots \binom{q-1}{a_1} \binom{q-2}{a_0} a^{N-A_{2h+1}} \neq 0. \end{aligned}$$

Therefore, if $r < \ell$, the polynomial $f(x)$ does not permute \mathbb{F}_{q^e} . Since $q-1 \mid N$, we can use Lemma 2.3 in the case $r \geq \ell$ and conclude the proof. \square

We now consider $r = \left[h \left(\sum_{i=0}^{e-3} q^i \right) \right] q + 1 = h \left(\sum_{i=1}^{e-2} q^i \right) + 1$ for $1 \leq h \leq q-1$. In this case, we obtain a nonexistence result by considering $N = 2q^{e-1} - q^{e-2} - 1$. The next result provides the exact elements that belong to S_N depending on h .

Lemma 4.4. Let $f(x) = x^r(x^{q-1}+a) \in \mathbb{F}_{q^e}[x]$ with $e, q \geq 3$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. Suppose $r = h(\ell - q^{e-1} - 1) + 1$ for some $1 \leq h \leq q - 1$. For $N = 2q^{e-1} - q^{e-2} - 1$ and $j = 2hq^{e-3} + h \sum_{i=0}^{e-4} q^i$, we have

$$\begin{aligned} \text{(i)} \quad & A_j = (h-2)q^{e-2} + (2h-1)q^{e-3} + (h-1) \sum_{i=0}^{e-4} q^i, \\ \text{(ii)} \quad & A_{j+1} = q^{e-1} + (h-1)q^{e-2} + 2hq^{e-3} + h \sum_{i=0}^{e-4} q^i, \\ \text{(iii)} \quad & S_N = \begin{cases} \{A_{j+1}\} & \text{if } h = 1 \\ \{A_j, A_{j+1}\} & \text{if } 2 \leq h \leq q-2 \\ \{A_j\} & \text{if } h = q-1. \end{cases} \end{aligned}$$

Proof. We start by showing (i) and (ii). By using that $N/(q-1) = 2q^{e-2} + \sum_{i=0}^{e-3} q^i$, we compute

$$\begin{aligned} A_j &= \left(2hq^{e-3} + h \sum_{i=0}^{e-4} q^i \right) \left(\sum_{i=0}^{e-1} q^i \right) - \left(h \left(\sum_{i=1}^{e-2} q^i \right) + 1 \right) \left(2q^{e-2} + \sum_{i=0}^{e-3} q^i \right) \\ &= 2h \sum_{i=e-3}^{2e-4} q^i + h \sum_{i=0}^{e-4} \sum_{j=0}^{e-1} q^{i+j} - 2h \sum_{i=e-1}^{2e-4} q^i - h \sum_{i=1}^{e-2} \sum_{j=0}^{e-3} q^{i+j} - 2q^{e-2} - \sum_{i=0}^{e-3} q^i \\ &= 2hq^{e-3} + 2(h-1)q^{e-2} - \sum_{i=0}^{e-3} q^i + h \sum_{i=0}^{e-1} q^i \\ &\quad + h \sum_{i=e}^{2e-5} q^i + h \sum_{i=e-1}^{2e-6} q^i - h \sum_{i=e-2}^{2e-5} q^i - h \sum_{i=e-3}^{2e-6} q^i \\ &= 2hq^{e-3} + 2(h-1)q^{e-2} - \sum_{i=0}^{e-3} q^i + h \sum_{i=0}^{e-1} q^i - hq^{e-1} - 2hq^{e-2} - hq^{e-3} \\ &= (h-2)q^{e-2} + hq^{e-3} + (h-1) \sum_{i=0}^{e-3} q^i \\ &= (h-2)q^{e-2} + (2h-1)q^{e-3} + (h-1) \sum_{i=0}^{e-4} q^i. \end{aligned}$$

We also compute

$$\begin{aligned} A_{j+1} &= A_j + \ell = A_j + \sum_{i=0}^{e-1} q^i \\ &= (h-2)q^{e-2} + (2h-1)q^{e-3} + (h-1) \sum_{i=0}^{e-4} q^i + \sum_{i=0}^{e-1} q^i \\ &= q^{e-1} + (h-1)q^{e-2} + 2hq^{e-3} + h \sum_{i=0}^{e-4} q^i. \end{aligned}$$

We now show (iii). By Lemma 2.2, since $\ell = \sum_{i=0}^{e-1} q^i$ and $N/\ell = (2q^{e-1} - q^{e-2} - 1)/\ell < 2$, we have $|S_N| \leq 2$. The elements in S_N satisfy $A_{i\pm 1} = A_i \pm \ell$; in particular, $A_i < A_k$ whenever $i < k$. Next we describe the elements in S_N depending on h .

Case 1: $h = 1$

Since $A_j = -q^{e-2} + q^{e-3}$ is negative, it does not belong to S_N . Let us consider A_{j+1} . We have $A_{j+1} = q^{e-1} + 2q^{e-3} + \sum_{i=0}^{e-4} q^i \geq 0$ and

$$\begin{aligned} A_{j+1} &\leq 2q^{e-1} - q^{e-2} - 1 = N \\ \iff 2q^{e-3} + \sum_{i=0}^{e-4} q^i &\leq q^{e-1} - q^{e-2} - 1 \\ \iff 2q^{e-3} + \sum_{i=0}^{e-4} q^i &\leq (q-1) \sum_{i=0}^{e-2} q^i - q^{e-2} \\ \iff 2q^{e-3} + \sum_{i=0}^{e-4} q^i &\leq (q-2)q^{e-2} + (q-1) \sum_{i=0}^{e-3} q^i, \end{aligned}$$

which is true, since $2q^{e-3} \leq (q-2)q^{e-2}$ and $\sum_{i=0}^{e-4} q^i \leq (q-1) \sum_{i=0}^{e-3} q^i$, for $q \geq 3$. Hence $A_{j+1} \in S_N$.

Since

$$A_{j+2} = A_{j+1} + \ell = 2q^{e-1} + q^{e-2} + 3q^{e-3} + 2 \sum_{i=0}^{e-4} q^i > N,$$

we conclude that $S_N = \{A_{j+1}\}$.

Case 2: $2 \leq h \leq q-2$

We have

$$\begin{aligned} 0 &\leq 3q^{e-3} + \sum_{i=0}^{e-4} q^i \\ &\leq A_j < A_{j+1} \\ &\leq 2q^{e-1} - q^{e-2} - 3q^{e-3} - \sum_{i=1}^{e-4} q^i - 2 \\ &\leq 2q^{e-1} - q^{e-2} - 1 = N, \end{aligned}$$

which shows that $S_N = \{A_j, A_{j+1}\}$.

Case 3: $h = q - 1$

In this case, we have $A_j = q^{e-1} - q^{e-2} - 2q^{e-3} - \sum_{i=0}^{e-4} q^i - 1$. So

$$A_{j-1} = A_j - \sum_{i=0}^{e-1} q^i = -2q^{e-2} - 3q^{e-3} - 2 \sum_{i=0}^{e-4} q^i - 1 < 0,$$

$$0 \leq A_j \leq 2q^{e-1} - q^{e-2} - 1 = N, \quad \text{and}$$

$$A_{j+1} = 2q^{e-1} - q^{e-3} - 1 > 2q^{e-1} - q^{e-2} - 1 = N.$$

Hence $S_N = \{A_j\}$. □

Proposition 4.5. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e, q \geq 3$, q a power of a prime p , $a \neq 0$, and $(-a)^\ell \neq 1$, and let $\ell = q^{e-1} + \dots + q + 1$. Suppose that $r \pmod{\ell} = h(\ell - q^{e-1} - 1) + 1$ for some $1 \leq h \leq q - 1$. Then $f(x)$ does not permute \mathbb{F}_{q^e} .*

Proof. By Proposition 4.3, if q is odd and $p \nmid h \left(\sum_{i=0}^{e-3} q^i \right)$, then $f(x)$ does not permute \mathbb{F}_{q^e} . Hence $p \mid h$, if q is odd. Assume $r < \ell$. We consider the following three cases: $h = 1$, $2 \leq h \leq q - 2$, and $h = q - 1$. Let $N = 2q^{e-1} - q^{e-2} - 1$ whose q -expansion is given by $q^{e-1} + (q-2)q^{e-2} + (q-1) \sum_{i=0}^{e-3} q^i$. Lemma 4.4 provides the exact elements in S_N in each case.

Case 1: $h = 1$

In this case, $S_N = \{A_{j+1}\}$ and $A_{j+1} = q^{e-1} + 2q^{e-3} + \sum_{i=0}^{e-4} q^i$ is the q -adic expansion of A_{j+1} . We compute $\sum_{A \in S_N} \binom{N}{A} a^{N-A}$

$$= \binom{N}{A_{j+1}} a^{N-A_{j+1}} = \binom{1}{1} \binom{q-2}{0} \binom{q-1}{2} \binom{q-1}{1}^{e-3} a^{N-A_{j+1}},$$

which is not zero by Lemma 2.5(i).

Case 2: $2 \leq h \leq q - 2$

We have $S_N = \{A_j, A_{j+1}\}$.

Subcase 2.1: $2 \leq h < \lfloor q/2 \rfloor$

In this case,

$$A_j = (h-2)q^{e-2} + (2h-1)q^{e-3} + (h-1) \sum_{i=0}^{e-4} q^i \quad \text{and}$$

$$A_{j+1} = q^{e-1} + (h-1)q^{e-2} + 2hq^{e-3} + h \sum_{i=0}^{e-4} q^i,$$

so

$$\begin{aligned} \sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_j} a^{N-A_j} + \binom{N}{A_{j+1}} a^{N-A_{j+1}} \\ &= \binom{1}{0} \binom{q-2}{h-2} \binom{q-1}{2h-1} \binom{q-1}{h-1}^{e-3} a^{N-A_j} \\ &\quad + \binom{1}{1} \binom{q-2}{h-1} \binom{q-1}{2h} \binom{q-1}{h}^{e-3} a^{N-A_{j+1}}. \end{aligned}$$

By Lemma 2.5(ii), when q is even, $\binom{q-2}{k} = 0$ if and only if k is odd. Also, if q is odd, $p \mid h$ and we get $\binom{q-2}{h-1} \equiv 0 \pmod{p}$. Therefore

$$\begin{aligned} &\sum_{A \in S_N} \binom{N}{A} a^{N-A} \\ &= \begin{cases} \binom{q-2}{h-1} \binom{q-1}{2h} \binom{q-1}{h}^{e-3} a^{N-A_{j+1}} \neq 0, & \text{if } h \text{ is odd and } q \text{ is even} \\ \binom{q-2}{h-2} \binom{q-1}{2h-1} \binom{q-1}{h-1}^{e-3} a^{N-A_j} \neq 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Subcase 2.2: q is even and $h = q/2$

In this case, the q -adic expansions of the two elements in S_N are

$$A_j = \left(\frac{q}{2} - 2\right)q^{e-2} + (q-1)q^{e-3} + \left(\frac{q}{2} - 1\right) \sum_{i=0}^{e-4} q^i \quad \text{and}$$

$$A_{j+1} = q^{e-1} + \left(\frac{q}{2}\right)q^{e-2} + \frac{q}{2} \sum_{i=0}^{e-4} q^i.$$

Consequently,

$$\begin{aligned}
\sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_j} a^{N-A_j} + \binom{N}{A_{j+1}} a^{N-A_{j+1}} \\
&= \binom{1}{0} \binom{q-2}{q/2-2} \binom{q-1}{q-1} \binom{q-1}{q/2-1}^{e-3} a^{N-A_{j+1}+\ell} \\
&\quad + \binom{1}{1} \binom{q-2}{q/2} \binom{q-1}{0} \binom{q-1}{q/2}^{e-3} a^{N-A_{j+1}} \\
&= \binom{q-2}{q/2} \binom{q-1}{q/2}^{e-3} a^{N-A_{j+1}} (a^\ell + 1).
\end{aligned}$$

Since ℓ is odd and $(-a)^\ell \neq 1$, we have $a^\ell + 1 \neq 0$. We conclude that the sum is not zero by applying Lemma 2.5(i) and (ii).

Subcase 2.3: $\lfloor q/2 \rfloor < h \leq q-2$

Write $h = \lfloor q/2 \rfloor + h_0$ with $h_0 > 0$.

If q is even, then $h = q/2 + h_0$, $0 < h_0 \leq (q-4)/2$ and the q -adic expansions of A_j and A_{j+1} are

$$\begin{aligned}
A_j &= (h-1)q^{e-2} + (2h_0-1)q^{e-3} + \sum_{k=0}^{e-4} (h-1)q^k \quad \text{and} \\
A_{j+1} &= q^{e-1} + hq^{e-2} + 2h_0q^{e-3} + h \sum_{k=0}^{e-4} q^k.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_j} a^{N-A_j} + \binom{N}{A_{j+1}} a^{N-A_{j+1}} \\
&= \binom{1}{0} \binom{q-2}{h-1} \binom{q-1}{2h_0-1} \binom{q-1}{h-1}^{e-3} a^{N-A_j} \\
&\quad + \binom{1}{1} \binom{q-2}{h} \binom{q-1}{2h_0} \binom{q-1}{h}^{e-3} a^{N-A_{j+1}} \\
&= \begin{cases} \binom{q-2}{h} \binom{q-1}{2h_0} \binom{q-1}{h}^{e-3} a^{N-A_{j+1}} \neq 0, & \text{if } h \text{ even} \\ \binom{q-2}{h-1} \binom{q-1}{2h_0-1} \binom{q-1}{h-1}^{e-3} a^{N-A_j} \neq 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

If q is odd, then $h = (q-1)/2 + h_0$ with $0 < h_0 \leq (q-3)/2$, and the q -adic expansions of A_j and A_{j+1} are

$$A_j = (h-1)q^{e-2} + (2h_0-2)q^{e-3} + (h-1) \sum_{i=0}^{e-4} q^i \quad \text{and}$$

$$A_{j+1} = q^{e-1} + hq^{e-2} + (2h_0-1)q^{e-3} + h \sum_{i=0}^{e-4} q^i.$$

This implies that

$$\begin{aligned} \sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_j} a^{N-A_j} + \binom{N}{A_{j+1}} a^{N-A_{j+1}} \\ &= \binom{1}{0} \binom{q-2}{h-1} \binom{q-1}{2h_0-2} \binom{q-1}{h-1}^{e-3} a^{N-A_j} \\ &\quad + \binom{1}{1} \binom{q-2}{h} \binom{q-1}{2h_0-1} \binom{q-1}{h}^{e-3} a^{N-A_{j+1}}. \end{aligned}$$

By using Lemma 2.5(i) and (ii), since $p \mid h$, we obtain that this sum is

$$\binom{q-2}{h} \binom{q-1}{2h_0-1} \binom{q-1}{h}^{e-3} a^{N-A_{j+1}} \neq 0.$$

Case 3: $h = q-1$

In this case, $S_N = \{A_j\}$ and the q -adic expansion of A_j is $(q-2)q^{e-2} + (q-3)q^{e-3} + (q-2) \sum_{i=0}^{e-4} q^i$.

Then

$$\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \binom{N}{A_j} a^{N-A_j} = \binom{1}{0} \binom{q-2}{q-2} \binom{q-1}{q-3} \binom{q-1}{q-2}^{e-3} a^{N-A_j},$$

which is not zero by Lemma 2.5(i).

Therefore, if $r < \ell$, the polynomial $f(x)$ does not permute \mathbb{F}_{q^e} . Since $N/(q-1) = 2q^{e-2} + \sum_{i=1}^{e-3} q^i$ is an integer, we use Lemma 2.3 for the case $r \geq \ell$ and conclude the proof. \square

5. Characterizations

Permutation binomials of the form $f(x) = x^r(x^{q-1}+a)$ over \mathbb{F}_{q^e} were studied by Hou when $e = 2$, and by Liu when $e = 3$ with q odd and $1 \leq r \leq \ell$, where $\ell = q^{e-1} + \dots + q + 1$; see Proposition 1.1.

In this section we prove Theorem 1.3 by applying our results from Sections 3 and 4 to present complete characterizations of these permutation binomials over \mathbb{F}_{q^2} , \mathbb{F}_{q^3} , \mathbb{F}_{q^4} , \mathbb{F}_{p^5} , and \mathbb{F}_{p^6} , where p is an odd prime, for all values of r . As we mentioned earlier, two necessary conditions are that $(-a)^\ell \neq 1$ and $\gcd(r, q-1) = 1$. We make use of these two facts without any further reference. In addition, as in Section 4, we show our results for $e \in \{5, 6\}$ by using Proposition 2.1 and Lucas' Theorem without referring to them.

5.1. Fields \mathbb{F}_{q^2} , \mathbb{F}_{q^3} , and \mathbb{F}_{q^4}

We extend Proposition 1.1 in the following directions: from $r \in \{1, \dots, \ell\}$ to $r \geq 1$, from q odd and $e = 3$ to any q and $e = 3$, and from $e \in \{2, 3\}$ to $e \in \{2, 3, 4\}$. Therefore our characterization provides a full description of permutation binomials $f(x) = x^r(x^{q-1} + a)$ over \mathbb{F}_{q^e} for $e \in \{2, 3, 4\}$. For these values of e , the description of r written in terms of ℓ turns out to be the same.

Proposition 5.1. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \in \{2, 3, 4\}$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. Then $f(x)$ permutes \mathbb{F}_{q^e} if and only if $(-a)^\ell \neq 1$, $\gcd(r, q-1) = 1$, and $r \pmod{\ell} \in \{1, \ell - q\}$.*

Proof. By Corollary 3.5, if $(-a)^\ell \neq 1$, $r \pmod{\ell} \in \{1, \ell - q\}$, and $\gcd(r, q-1) = 1$, then $f(x)$ permutes \mathbb{F}_{q^e} . Conversely, suppose that $f(x)$ permutes \mathbb{F}_{q^e} for $e \in \{2, 3, 4\}$. By Proposition 4.1, $r \pmod{\ell} = hq + 1$ for some integer h .

When $e = 2$, since $\ell = q + 1$, we must have $r \pmod{\ell} = 1$.

When $e = 3$, we have $\ell = (q+1)q + 1$, so $0 \leq h \leq q$. Proposition 4.5 implies that $h \in \{0, q\}$, so $r \pmod{\ell} \in \{1, \ell - q\}$.

When $e = 4$, we have $\ell = (q^2 + q + 1)q + 1$. Combining Propositions 4.1 and 4.2 gives that $r \pmod{\ell} = k(q+1)q + 1$ with $0 \leq k \leq q$. Proposition 4.5 implies that $k \in \{0, q\}$, so $r \pmod{\ell} \in \{1, \ell - q\}$. \square

For fields \mathbb{F}_{q^2} , we note that $r \pmod{\ell} = \ell - q = 1$.

5.2. Fields \mathbb{F}_{p^5} , p an odd prime

In Proposition 5.1 we show that the only values of $r \pmod{\ell}$ such that $x^r(x^{q-1} + a)$ permutes \mathbb{F}_{q^e} for $e \in \{2, 3, 4\}$ and $a \neq 0$ are 1 and $\ell - q$, with $\gcd(r, q-1) = 1$. In $\mathbb{F}_{p^5}[x]$ there are two additional values.

Proposition 5.2. *Let $f(x) = x^r(x^{p-1} + a) \in \mathbb{F}_{p^5}[x]$ with p an odd prime and $a \neq 0$, and let $\ell = p^4 + p^3 + p^2 + p + 1$. Then $f(x)$ permutes \mathbb{F}_{p^5} if and only if $(-a)^\ell \neq 1$, $\gcd(r, p-1) = 1$, and $r \pmod{\ell} \in \{1, p^3 + 1, p^4 + p^2 + 1, \ell - p\}$.*

Proof. By Corollary 3.5, if $(-a)^\ell \neq 1$, $r \pmod{\ell} \in \{1, p^3 + 1, p^4 + p^2 + 1, \ell - p\}$, and $\gcd(r, p-1) = 1$, then $f(x)$ permutes \mathbb{F}_{p^5} . Conversely, let $f(x)$ be a permutation binomial over \mathbb{F}_{p^5} . Proposition 4.1 and Proposition 4.3 imply that $r \pmod{\ell}$ has the form $hp^2 + 1$ for some $0 \leq h \leq p^2 + p + 1$. We show that we cannot have the following three cases for h : $1 \leq h \leq p-1$, $p+1 \leq h \leq p^2$, and $p^2 + 2 \leq h \leq p^2 + p$.

Case 1: $1 \leq h \leq p-1$

Consider $N = p^4 - p^2 - p + 1$ whose p -adic expansion is $(p-1)p^3 + (p-2)p^2 + (p-1)p + 1$. By Lemma 2.2, since $N < \ell$, the set S_N has at most one element. We compute

$$\begin{aligned} A_{hp} &= hp(p^4 + p^3 + p^2 + p + 1) - \frac{(hp^2 + 1)(p^4 - p^2 - p + 1)}{p-1} \\ &= hp(p^4 + p^3 + p^2 + p + 1) - (hp^2 + 1)(p^3 + p^2 - 1) \\ &= (h-1)p^3 + (2h-1)p^2 + hp + 1. \end{aligned}$$

Since $1 \leq h \leq p-1$, we have $0 < A_{hp} \leq p^4 - 2p^2 - p + 1 < N$. Hence A_{hp} is the unique element in S_N .

Subcase 1.1: $1 \leq h \leq (p-1)/2$

By using Lucas' Theorem and Lemma 2.5(i) and (ii), we have

$$\begin{aligned} \sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_{hp}} a^{N-A_{hp}} \\ &= \binom{p-1}{h-1} \binom{p-2}{2h-1} \binom{p-1}{h} \binom{1}{1} a^{N-A_{hp}} \neq 0. \end{aligned}$$

Subcase 1.2: $(p-1)/2 < h \leq p-1$

Write $h = (p-1)/2 + h_0$ with $1 \leq h_0 \leq (p-1)/2$. Then

$$\begin{aligned} A_{hp} &= (h-1)p^3 + \left(2 \left(\frac{p-1}{2} + h_0\right) - 1\right)p^2 + hp + 1 \\ &= hp^3 + (2h_0 - 2)p^2 + hp + 1. \end{aligned}$$

Since $0 \leq 2h_0 - 2 \leq p-3$, we have

$$\begin{aligned}
\sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_{hp}} a^{N-A_{hp}} \\
&= \binom{p-1}{h} \binom{p-2}{2h_0-2} \binom{p-1}{h} \binom{1}{1} a^{N-A_{hp}} \neq 0.
\end{aligned}$$

Case 2: $p+1 \leq h \leq p^2$

Consider $N = p^3 - 1$ whose p -adic expansion is $(p-1)p^2 + (p-1)p + p - 1$. By Lemma 2.2, since $N < \ell$, the set S_N has at most one element. We compute

$$\begin{aligned}
A_h &= h(p^4 + p^3 + p^2 + p + 1) - \frac{(hp^2 + 1)(p^3 - 1)}{p-1} \\
&= h(p^4 + p^3 + p^2 + p + 1) - (hp^2 + 1)(p^2 + p + 1) \\
&= hp + h - p^2 - p - 1.
\end{aligned}$$

Since $p+1 \leq h \leq p^2$, we have $0 < A_h \leq p^3 - p - 1 < N$, so A_h is the unique element in S_N . Let $A_h = a_2p^2 + a_1p + a_0$ be the p -adic expansion of A_h . Since $a_0, a_1, a_2 \leq p-1$, Lucas' Theorem and Lemma 2.5(i) imply that

$$\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \binom{N}{A_h} a^{N-A_h} = \binom{p-1}{a_2} \binom{p-1}{a_1} \binom{p-1}{a_0} a^{N-A_h} \neq 0.$$

Case 3: $p^2 + 2 \leq h \leq p^2 + p$

As in Case 1, consider $N = p^4 - p^2 - p + 1 = (p-1)p^3 + (p-2)p^2 + (p-1)p + 1$. Then S_N has at most one element. We compute

$$\begin{aligned}
A_{hp-p-1} &= (hp - p - 1)(p^4 + p^3 + p^2 + p + 1) - \frac{(hp^2 + 1)(p^4 - p^2 - p + 1)}{p-1} \\
&= (hp - p - 1)(p^4 + p^3 + p^2 + p + 1) - (hp^2 + 1)(p^3 + p^2 - 1) \\
&= -p^5 - 2p^4 + (h-3)p^3 + (2h-3)p^2 + (h-2)p.
\end{aligned}$$

For $p^2 + 2 \leq h \leq p^2 + p$, we have $0 < p^2 \leq A_{hp-p-1} \leq p^4 - 2p^2 - 2p < N$, so A_{hp-p-1} is the unique element in S_N . Write $h = p^2 + 2 + h_0$ with $0 \leq h_0 \leq p-2$. Then

$$A_{hp-p-1} = h_0p^3 + (2h_0 + 1)p^2 + h_0p.$$

Subcase 3.1: $0 \leq h_0 \leq (p-3)/2$

Since $2h_0 + 1 \leq p - 2$, it follows that

$$\begin{aligned} \sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_{hp-p-1}} a^{N-A_{hp-p-1}} \\ &= \binom{p-1}{h_0} \binom{p-2}{2h_0+1} \binom{p-1}{h_0} \binom{1}{0} a^{N-A_{hp-p-1}} \neq 0. \end{aligned}$$

Subcase 3.2: $(p-3)/2 < h_0 \leq p-2$

Write $h_0 = (p-3)/2 + h_1$ with $1 \leq h_1 \leq (p-1)/2$. Then

$$A_{hp-p-1} = (h_0 + 1)p^3 + (2h_1 - 2)p^2 + h_0p.$$

and, since $2h_1 - 2 \leq p - 3$, we have

$$\begin{aligned} \sum_{A \in S_N} \binom{N}{A} a^{N-A} &= \binom{N}{A_{hp-p-1}} a^{N-A_{hp-p-1}} \\ &= \binom{p-1}{h_0+1} \binom{p-2}{2h_1-2} \binom{p-1}{h_0} \binom{1}{0} a^{N-A_{hp-p-1}} \neq 0. \end{aligned}$$

By Proposition 2.1, $f(x)$ cannot permute \mathbb{F}_{p^5} with $r \pmod{\ell} = hp^2 + 1$ for $1 \leq h \leq p-1$, $p+1 \leq h \leq p^2$, and $p^2+2 \leq h \leq p^2+p$, and this contradicts our assumption. The remaining values of h are $0, p, p^2+1, p^2+p+1$, hence necessarily $r \pmod{\ell} \in \{1, p^3+1, p^4+p^2+1, \ell-p\}$.

This completes the proof. \square

5.3. Fields \mathbb{F}_{p^6} , p an odd prime

The characterization of permutation binomials of the form $f(x) = x^r(x^{p-1} + a)$ over \mathbb{F}_{p^6} is the same one as for \mathbb{F}_{q^2} , \mathbb{F}_{q^3} , and \mathbb{F}_{q^4} from Proposition 5.1.

Proposition 5.3. *Let $f(x) = x^r(x^{p-1} + a) \in \mathbb{F}_{p^6}[x]$ with p an odd prime and $a \neq 0$, and let $\ell = p^5 + p^4 + p^3 + p^2 + p + 1$. Then $f(x)$ permutes \mathbb{F}_{p^6} if and only if $(-a)^\ell \neq 1$, $\gcd(r, p-1) = 1$, and $r \pmod{\ell} \in \{1, \ell-p\}$.*

Proof. By Corollary 3.5, if $(-a)^\ell \neq 1$, $r \pmod{\ell} \in \{1, \ell-p\}$, and $\gcd(r, p-1) = 1$, then $f(x)$ permutes \mathbb{F}_{p^6} . Conversely, suppose that $f(x)$ permutes \mathbb{F}_{p^6} . By combining Propositions 4.1-4.3, we may assume $r \pmod{\ell} = n(p^3 + p^2) + 1$ with $0 \leq n \leq p^2 + 1$.

We will show that we cannot have $1 \leq n \leq p^2$. Let $N = p^5 - p^3 + p^2 - 1$ with p -expansion given by $(p-1)p^4 + (p-1)p^3 + (p-1)p + p - 1$. Then $\lfloor N/\ell \rfloor = 0$. By Lemma 2.2, $|S_N| \leq 1$. Let

$j = n(p^2 + p - 1) + 1$. We compute A_j

$$\begin{aligned}
&= [n(p^2 + p - 1) + 1]\ell - \frac{[n(p^3 + p^2) + 1](p^5 - p^3 + p^2 - 1)}{p - 1} \\
&= [n(p^2 + p - 1) + 1]\ell - (np^3 + np^2 + 1)(\ell - p^5 - p^2) \\
&= np^2\ell + npl - n\ell + \ell - np^3\ell + np^8 + np^5 - np^2\ell + np^7 + np^4 - \ell + p^5 + p^2 \\
&= n[p\ell - \ell + p^3(-\ell + p^5 + p^4 + p^2 + p)] + p^5 + p^2 \\
&= n(-p^3 - 1) + p^5 + p^2 \\
&= p^5 - np^3 + p^2 - n.
\end{aligned}$$

Since $1 \leq n \leq p^2$, we have $0 \leq A_j \leq N$, so $S_N = \{A_j\}$.

Case 1: $1 \leq n < p^2$

Write $n = n_1p + n_0$, where $0 \leq n_0, n_1 \leq p - 1$. If $n_0 \neq 0$, the p -expansion of A_j is

$$A_j = (p - n_1 - 1)p^4 + (p - n_0)p^3 + (p - n_1 - 1)p + (p - n_0).$$

By Lemma 2.5(i), it turns out that

$$\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \binom{p-1}{p-1-n_1} \binom{p-1}{p-n_0} \binom{p-1}{p-1-n_1} \binom{p-1}{p-n_0} a^{N-A_j} \neq 0.$$

If $n_0 = 0$, then $n_1 \neq 0$ and $A_j = (p - n_1)p^4 + (p - n_1)p$, where $1 \leq p - n_1 \leq p - 1$. Thus,

$$\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \binom{p-1}{p-n_1} \binom{p-1}{p-n_1} a^{N-A_j} \neq 0.$$

Case 2: $n = p^2$

Then $A_j = 0$ and $\sum_{A \in S_N} \binom{N}{A} a^{N-A} = \binom{N}{0} a^N \neq 0$.

By Proposition 2.1, $f(x)$ cannot permute \mathbb{F}_{p^6} with $r \pmod{\ell} = n(p^3 + p^2) + 1$, $1 \leq n \leq p^2$, and this contradicts our assumption. Hence $n \in \{0, p^2 + 1\}$ and $r \pmod{\ell} \in \{1, \ell - p\}$. \square

5.4. Fields \mathbb{F}_{q^e}

We conjecture that all permutations binomials of the form $x^r (x^{q-1} + a)$ over \mathbb{F}_{q^e} are the ones described in Theorem 3.4. More specifically, we have the following.

Conjecture 5.4. *Let $f(x) = x^r(x^{q-1} + a) \in \mathbb{F}_{q^e}[x]$ with $e \geq 2$ and $a \neq 0$, and let $\ell = q^{e-1} + \dots + q + 1$. Then $f(x)$ permutes \mathbb{F}_{q^e} if and only if $f(x)$ is congruent to the composition of a linearized binomial $L(x) = x^{q^h} + ax$ and the monomial x^r modulo $x^{q^e} - x$, where $(-a)^\ell \neq 1$ and $\gcd(r, q-1) = 1$.*

If Conjecture 5.4 holds true, by Theorem 3.4 the only values of r for which $f(x) = x^r(x^{q-1} + a)$ permutes \mathbb{F}_{q^e} satisfy $r = s\ell + \sum_{i=0}^{k-1} q^{hi} \pmod{q^e - 1}$, where $\gcd(h, e) = 1$, $k \pmod{e} = h^{-1}$, and s is an integer. The results in Sections 5.1 to 5.3 confirm Conjecture 5.4 for fields \mathbb{F}_{q^e} , with $e = 2, 3, 4$, and \mathbb{F}_{p^e} , with $e = 5, 6$ and an odd prime p . An exhaustive search for all permutation binomials of the form $f(x)$ for $q^e < 10^8$ also verified the conjecture. To optimize our search, we used results from Sections 3 to 5 along with the following.

Proposition 5.5. *Let ξ be a primitive element of $\mathbb{F}_{q^e}^*$, and $f_m(x) = x^r(x^{q-1} + \xi^m) \in \mathbb{F}_{q^e}[x]$. If $f_j(x)$ does not permute \mathbb{F}_{q^e} for all $0 \leq j < q-1$, then $f_m(x)$ does not permute \mathbb{F}_{q^e} for all $0 \leq m < q^e - 1$.*

Proof. Let $0 \leq m < q^e - 1$ and write $m = s(q-1) + j$ with $0 \leq j < q-1$. Since $f_j(x)$ does not permute \mathbb{F}_{q^e} , there exist integers i_0 and i_1 such that $0 \leq i_0 < i_1 < q^e - 1$ and $f_j(\xi^{i_0}) = f_j(\xi^{i_1})$. Thus, $\xi^{rs+s(q-1)} f_j(\xi^{i_0}) = \xi^{rs+s(q-1)} f_j(\xi^{i_1})$ implies that $\xi^{r(i_0+s)} (\xi^{(i_0+s)(q-1)} + \xi^m) = \xi^{r(i_1+s)} (\xi^{(i_1+s)(q-1)} + \xi^m)$. Therefore, $f_m(\xi^{i_0+s}) = f_m(\xi^{i_1+s})$ and $f_m(x)$ does not permute \mathbb{F}_{q^e} . \square

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References

- [1] C. Hermite, Sur les fonctions de sept lettres, C. R. Acad. Sci. Paris 57 (1863) 750–757 (1863).
- [2] X. Hou, Permutation polynomials over finite fields – a survey of recent advances, Finite Fields Appl. 32 (2015) 82–119 (2015).
- [3] X. Hou, A survey of permutation binomials and trinomials over finite fields, in: Topics in finite fields, Vol. 632 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2015, pp. 177–191 (2015).
- [4] G. L. Mullen, D. Panario (Eds.), Handbook of finite fields, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2013 (2013).
- [5] A. Akbary, D. Ghioca, Q. Wang, On permutation polynomials of prescribed shape, Finite Fields Appl. 15 (2) (2009) 195–206 (2009).
- [6] Q. Wang, Polynomials over finite fields: an index approach, in the Proceedings of Pseudo-Randomness and Finite Fields, Multivariate Algorithms and their Foundations in Number Theory, October 15-19, Linz, 2018, Combinatorics and Finite Fields. Difference Sets, Polynomials, Pseudorandomness and Applications (2019) 319–348 (2019).
- [7] A. M. Masuda, M. E. Zieve, Permutation binomials over finite fields, Trans. Amer. Math. Soc. 361 (8) (2009) 4169–4180 (2009).
- [8] X. Hou, Permutation polynomials of the form $x^r(a + x^{2(q-1)})$ – a nonexistence result, arXiv:1609.03662 (2016).
- [9] X. Hou, A class of permutation binomials over finite fields, J. Number Theory 133 (10) (2013) 3549–3558 (2013).
- [10] X. Hou, Permutation polynomials of \mathbb{F}_{q^2} of the form $aX + X^{r(q-1)+1}$, in: Contemporary developments in finite fields and applications, World Sci. Publ., Hackensack, NJ, 2016, pp. 74–101 (2016).
- [11] X. Hou, S. D. Lappano, Determination of a type of permutation binomials over finite fields, J. Number Theory 147 (2015) 14–23 (2015).

- [12] S. D. Lappano, A note regarding permutation binomials over \mathbb{F}_{q^2} , *Finite Fields Appl.* 34 (2015) 153–160 (2015).
- [13] M. E. Zieve, Permutation polynomials on \mathbb{F}_q induced from bijective Rédei functions on subgroups of the multiplicative group of \mathbb{F}_q , arXiv:1310.0776 (2013).
- [14] X. Liu, Some results about permutation properties of a kind of binomials over finite fields, arXiv:1906.09168 (2019).
- [15] K. Li, L. Qu, X. Chen, New classes of permutation binomials and permutation trinomials over finite fields, *Finite Fields Appl.* 43 (2017) 69–85 (2017).
- [16] A. Masuda, D. Panario, Q. Wang, The number of permutation binomials over \mathbb{F}_{4p+1} where p and $4p+1$ are primes, *Electron. J. Combin.* 13 (1) (2006) Research Paper 65, 15 (2006).