Study of r-Orthogonality for Latin Squares

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Abstract

A **Latin square** (LS) of order n , is an $n \times n$ array of n different elements, where in each row and each column the elements are never repeated. Latin squares have various applications in Coding Theory and Cryptography. The famous Sudoku squares are examples of latin squares. Two latin squares of order n are said to be **r-orthogonal** if when the squares are superimposed we get r distinct ordered pairs of symbols. In this work we study generalizations of r-orthogonality to sets of LS. Also, we will present some preliminary results on some of the properties of these generalizations.

1 Introduction

To understand and take advantage of the paper one will need to know some definitions and theorems. We start by formally defining the main concepts and some important theorems about latin squares. A **latin square** (LS) of order n is an $n \times n$ array containing n distinct symbols, usually denoted by $0, 1, \ldots, n-1$, with the characteristic that each row and column of the array contains each symbol exactly once (see Example 1).

Example 1. *All the latin squares of order* 3.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$E = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \qquad F = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \qquad K = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

To study the r-orthogonality of latin squares one must consider two special cases of latin squares which are **reduced latin squares** and **semi-reduced latin squares**. These are important to obtain subsets that can be used to study r-orthogonality without having to examine each latin square of order n. A **reduced latin square** is a latin square that has the first row and the first column in the standard order $0, 1, 2, \ldots, n-1$ In Example 1, array A shows the only reduced latin square of order 3. The symbol RLS will be used when referring to reduced latin squares.

The **semi-reduced latin square** is a latin square that has only the first row in the standard order $0, 1, 2, \ldots, n-1$ In Example 1, arrays A and B show the only standard latin square of order 3. The symbol SLS will be used when referring to semi-reduced latin squares.

The next theorem shows how to compute the numbers of latin squares of order n, knowing the number of RLS. This theorem is Theorem 2.2.6 proved in [1].

Theorem 1. *For any $n \geq 2$, $L_n = n!(n-1)!l_n$. Where L_n denotes the number of all latin squares of order n and l_n denotes the number of all reduced latin squares of order n.*

Now we define some concepts that are needed to understand r-orthogonality. Two latin squares A and B are **superimposed** when putting one on top of the other and forming the pairs $(\alpha_{(i,j)}, \beta_{(i,j)})$ where $\alpha_{(i,j)}$ is a element in the row i and column j from the latin square A and $\beta_{(i,j)}$ is a element in the row i and column j from the latin square B (see Example 2). The notation $\mathbf{N}(A, B)$ is used for the number of distinct order pairs and $\mathbf{S}(A, B)$ denotes the array of ordered pairs $(\alpha_{(i,j)}, \beta_{(i,j)})$.

Example 2. *Consider two latin squares A and B of order 3.*

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

Then

$$S(A,B) = \begin{pmatrix} (0,0) & (1,1) & (2,2) \\ (1,2) & (2,0) & (0,1) \\ (2,1) & (0,2) & (1,0) \end{pmatrix}$$

and $N(A,B) = 9$*. Also consider the arrays* A *and* C *in Example 1 that has* $N(A,C) = 3$

Two latin squares of order n are **r-orthogonal** if when the squares are superimposed we get r distinct ordered pairs of symbols. In the Example 2 and Example 3 the set of latin squares is 9-orthogonal.

Example 3. *The following latin squares* LS_x *and* LS_y *are 9-orthogonal*

$$LS_x = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \quad LS_y = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{pmatrix}$$

$$S(LS_x, LS_y) = \begin{pmatrix} (0,0) & (1,1) & (2,2) & (3,3) \\ (1,1)^* & (3,2) & (0,3) & (2,0) \\ (2,2)^* & (0,3)^* & (3,0) & (1,1)^* \\ (3,3)^* & (2,0)^* & (1,1)^* & (0,2) \end{pmatrix}$$

the pairs with the asterisk (*) are the pairs that are repeated in $S(A,B)$*.*

Two latin squares A and B of order n are said to be **orthogonal** if $N(A,B) = n^2$. For example, the arrays in Example 2 are orthogonal.

When every pair of LS in a set of two or more LS $S = \{LS_1, LS_2, \ldots, LS_t\}$ $t \geq 2$ are orthogonal, we say that the LS in S are **mutually orthogonal**.

To define the concept of frecuency of r - orthogonality of a set of LS we need more notation. Let $\{LS_1, \ldots, LS_t\}$ be a set of LS. Define

r_t - orthogonality =

$$\sum_{i=1}^{t-1} \sum_{j=i}^{t-1} N(LS_i, LS_{j+1})$$

where two LS are superimposed each time.

Taking this into consideration we define the **frequency** of r_t - orthogonality is the number of sets of t latins squares of order n with r_t - orthogonality is denote by $h_{r_t}(n)$.

Example 4. *$h_{6_2}(4) = 12$, and $h_{5_2}(5) = 56$ denote the frequency of two different cases of LS.*

Once the frequencies are found, they are clasified in tables to analyse how frequency changes depending on the r-orthogonality.

2 r- Orthogonality considering only set of RLS and SLS

To study the r-orthogonality and the frequency of r-orthogonality we need to make a lot of computations, as known in the Section 3 were we talk about some proposition between different sets of latin squares, some studies show that if we have a $N(LS_i, LS_j) = r$ where LS_i and LS_j are normal latin squares then exist $N(RLS_a, SLS_b) = r$ see Example 5 and Proposition 2.1. Knowing this the different r-orthogonality that have a set of normal latin squares $LS \times LS$ are going to be found in a set of reduce latin squares and semi-reduce latin squares $RLS \times SLS$. This results are important because the time of computations studying the set of $LS \times LS$ is more than studying the set of $RLS \times SLS$. For example the numbers of comparisons for computing the maximum r-orthogonality in a set of $LS \times LS$ of order 5 is $L_5 \times L_5$ that is $(5! \times 4! \times 56) \times (5! \times 4! \times 56) = 26011238400$ but if we do the same using the set of $RLS \times SLS$ the number of comparisons is $l_5 \times sl_5$ that is $(4! \times 56) \times 56 = 75264$ note this 0.000289352% of all LS of order 5.

Proposition 2.1. *To know every possible r-orthogonlity between any two Latin Squares of order n, we only need to superimpose every RLS with every SLS of said order.*

Proof. Consider any two non-standard LS of order n and r-orthogonality X. We can apply a relabeling function on each of the LS to transform then into SLS while preserving their X-orthogonality. Furthermore, by permuting the rows of one of the SLS we can obtain a RLS and applying that same permutation on the other SLS we get different SLS. At this point we still get the same r-orthogonality X. And so it follows that it is enough to superimpose every RLS with every SLS of a given order n to get every possible r-orthogonality between any LS of that order. □

Example 5. *Take two non-standard latin squares LS_a and LS_b*

$$LS_a = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 1 & 2 & 0 & 3 \end{pmatrix} \qquad LS_b = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}$$

Doing the superimposition of LS_a and LS_b we will get the following matrix and $N(LS_a, LS_b) = 12$.

$$S(LS_a, LS_b) = \begin{pmatrix} (2,1) & (1,0) & (3,2) & (0,3) \\ (3,0) & (0,3) & (2,1) & (1,2) \\ (0,2) & (3,1) & (1,3) & (2,0) \\ (1,3) & (2,2) & (0,0) & (3,1) \end{pmatrix}$$

Applying the relabel permutations $\rho_a = \binom{2\ 1\ 3\ 0}{0\ 1\ 2\ 3}$ to the latin square LS_a and the permutation $\rho_b = \binom{1\ 0\ 2\ 3}{0\ 1\ 2\ 3}$ to the latin square LS_b we will get the latin squares LS_a and LS_b in standard form.

$$SLS_a = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{pmatrix} \qquad SLS_b = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

Where SLS_a and SLS_b denote the standard forms of LS_a and LS_b respectively.
Then applying the row permutation $\rho_{row} = \binom{0\ 1\ 2\ 3}{0\ 2\ 3\ 1}$ to the latin squares SLS_a and SLS_b. Observe that SLS_a is transformed in to a reduced latin square.

$$RLS_a \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \qquad SLS_b^* \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \end{pmatrix}$$

Where RLS_a denotes the reduced from of SLS_a. If you made the the superimpositions of RLS_a and SLS_b^* you will get the following matrix and $N(RLS_a, SLS_b^*) = 12$.

$$S(RLS_a, SLS_b^*) = \begin{pmatrix} (0,0) & (1,1) & (2,2) & (3,3) \\ (1,3) & (0,2) & (3,1) & (2,0) \\ (2,1) & (3,3) & (0,0) & (1,2) \\ (3,2) & (2,0) & (1,3) & (0,1) \end{pmatrix}$$

2.1 Algorithm for computing frequency for latin squares of order n.

The following pseudocode gives the frequency table for the $r_t - orthogonality$ of latin squares of order n. The algorithm requires to especify the value of n and also to give every semi-reduced latin square of said order as input. As a result we get two output files, one with the frequency table and another containing all the sets of t latin squares that give the maximum $r_t - orthogonality$. First we declare some constant values:

- $n =$ the order of the Latin Square.
- $reds =$ the number of reduced Latin Squares of order n.
- $NUMLS =$ the total of semi-reduced Latin Squares of order n.
- $minM = n \times \binom{t}{2}$; minimum possible $r_t - orthogonality$.
- $maxM = n^2 \times \binom{t}{2}$; maximum possible $r_t - orthogonality$.
- $rs = maxM - minM + 1$; number of possible distinct $r_t - orthogonalities$.

2.2 Main program

$LS[NUMLS] = \{(Semi-reducedLatinSquaresofordern)\}$

```
for RLS = 0 to reds − 1 by 1 do
    for SLS_1 = 0 to NUMLS − 1 by 1 do
        for SLS_2 = SLS_1 to NUMLS − 1 by 1 do
                    ⋮                        ⋮
          for SLS_{t−1} = SLS_{t−2} to NUMLS − 1 by 1 do
            r_t = MAX(LS[RLS], LS[SLS_1], LS[SLS_2], . . . , LS[SLS_{t−1}])
                if(Max ≤ r_t) then
                    if(Max < r_t) then
                        Max = r_t
                        clear output file
                    print set of Latin Squares and its r-orthogonality
                increase freq[r_t − minM] by 1
close output file
print frequency
```

2.3 MAX function

Recieves t latin squares: $Lsqr_1, Lsqr_2, \ldots, Lsqr_t$

Return $\sum_{i=1}^{t-1}(\sum_{j=i}^{t-1} supimp(Lsqr_i, Lsqr_{j+1}))$

2.4 r-Orthogonality function

Recieves two Latin Squares to be superimposed: $Lsqr_1$ and $Lsqr_2$

$supimp[n^2] = \{0\}$
for $row = 0$ to $n^2 - 1$ by 1 do
    for $col = 0$ to $n^2 - 1$ by 1 do
        $supimp[Lsqr_1[row][col] \times n + Lsqr_2[row][col]] = 1$
$r - ort = 0$
for $s = 0$ to $n^2 - 1$ by 1 do
    if $(supimp[s] = 1)$ then
        increase r-ort by 1

Return r-ort

3 r- Orthogonality relation between sets of latin squares

The study of the r - orthogonality of latin squares is based in observe an analyze the superimposing of latin squares. Exist six different set to study the r - orthogonality of latin squares the first case is made the superimposing of a LS with another LS but in this we need to make to much superimposing. Another set is made the superimposing of one RLS and a SLS this set need less superimposing that the first set also this is the case that we use more in the investigation. We observed that knowing the r - orthogonality in one set we can know the same r - orthogonality in the other set. Next in the paper we show all the cases described and how move between set and their proof.

Using the theorem 1 we can formulate a corollary to know the the number of all semi-reduce latin square of order n denote sl_n.

Corollary 1. *For any $n \geq 2$, $sl_n = (n-1)!l_n$.*

Table 1: Latin square sets

Sets	Description
$LS \times LS$	Two Latin Squares are superimposed. This procedure is repeated for all Latin Square combinations.
$SLS \times RLS$	One Standar Latin Square and one Reduced Latin Square are superimposed. This procedure is repeated for all Latin Square combinations.
$SLS \times SLS$	Two Standar Latin Squares are superimposed. This procedure is repeated for all Latin Square combinations.
$LS \times RLS$	One Latin Square and one Reduced Latin Square are superimposed. This procedure is repeated for all Latin Square combinations.
$LS \times SLS$	One Latin Square and one Standar Latin Square are superimposed. This procedure is repeated for all Latin Square combinations.
$RLS \times RLS$	Two Reduced Latin Squares are superimposed. This procedure is repeated for all Latin Square combinations.

Proof. Given a RLS of order n, note that have n rows and remember the definition of a semi-reduce latin square, that have the first row in the standard order, now it is easy to see that by permuting the second row to the n row of a RLS the result is a SLS, knowing this note that by permuting the $(n-1)$ row we can generate $(n-1)!$ SLS from a RLS. Moreover note that all the SLS of order n is $(n-1)!l_n$. □

We need to define some notations to show how to know the r - orthogonality of a set of LS knowing the r - orthogonality of a different set.

Definition 1. $N_r(\eth)$ *represent the number of latin squares r - orthogonal in the set* $\eth$*. Moreover* $\eth$ *is a set from the table 1*

Example 6. $N_r(LS \times LS)$ *represent the number of latin squares r - orthogonal in the set* $LS \times LS$*. Also* $N_r(SLS \times RLS)$ *represent the number of latin squares r - orthogonal in the set* $SLS \times RLS$*.*

Knowing the theorem 1 and the corollary 1 and using the notations from the definition 1 consider next propositions.

Proposition 3.1. $N_r(SLS \times RLS) = \frac{LS \times LS}{(n-1)!(n!)^2}$

Proof. Since the number of SLS is $(n-1)!l_n$ by the theorem 1 and the number of RLS is l_n we know that $SLS \times RLS = (n-1)!(l_n)^2$. Moreover the numbers of LS is $n!(n-1)!l_n$ by the theorem 1 this show that $LS \times LS = (n!(n-1)!l_n)^2$ note following:

$$SLS \times RLS = \frac{LS \times LS}{(n-1)!(n!)^2} \tag{1}$$
$$= \frac{(n!(n-1)!l_n)^2}{(n-1)!(n!)^2} \tag{2}$$
$$= \frac{(n!(n-1)!l_n)^2}{(n-1)!(n!)^2} \tag{3}$$
$$= \frac{(n!(n-1)!l_n) \cdot (n!(n-1)!l_n)}{(n-1)!(n!)^2} \tag{4}$$
$$= \frac{(n!)^2((n-1)!)^2(l_n)^2}{(n-1)!(n!)^2} \tag{5}$$
$$\tag{6}$$

cancel the same term from the numerator and denominator show that $SLS \times RLS = (n-1)!(l_n)^2$ this prove that $SLS \times RLS = \frac{LS \times LS}{(n-1)!(n!)^2}$. □

Proposition 3.2. $N_r(SLS \times SLS) = \frac{LS \times LS}{(n!)^2}$

Proof. Since the number of SLS is $(n-1)!l_n$ by the theorem 1 we know that $SLS \times SLS = ((n-1)!(l_n))^2$. Moreover the numbers of LS is $n!(n-1)!l_n$ by the theorem 1 this show that $LS \times LS = (n!(n-1)!l_n)^2$ note following:

$$SLS \times SLS = \frac{LS \times LS}{(n!)^2} \tag{7}$$
$$= \frac{(n!(n-1)!l_n)^2}{(n!)^2} \tag{8}$$
$$= \frac{(n!(n-1)!l_n)^2}{(n!)^2} \tag{9}$$
$$= \frac{(n!(n-1)!l_n) \cdot (n!(n-1)!l_n)}{(n!)^2} \tag{10}$$
$$= \frac{(n!)^2((n-1)!)^2(l_n)^2}{(n!)^2} \tag{11}$$
$$\tag{12}$$

cancel the same term from the numerator and denominator show that $SLS \times SLS = ((n-1)!(l_n))^2$ this prove that $SLS \times SLS = \frac{LS \times LS}{(n!)^2}$. □

Proposition 3.3. $N_r(LS \times RLS) = \frac{LS \times LS}{(n-1)!(n!)}$

Proof. Since the number of LS is $(n!(n-1)!l_n)^2$ by the theorem 1 and the number of RLS is l_n we know that $LS \times RLS = n!(n-1)!(l_n)^2$. Moreover we konw that $LS \times LS = (n!(n-1)!l_n)^2$ note following:

$$LS \times RLS = \frac{LS \times LS}{(n-1)!(n!)} \quad (13)$$
$$= \frac{(n!(n-1)!l_n)^2}{(n-1)!(n!)} \quad (14)$$
$$= \frac{(n!(n-1)!l_n)^2}{(n-1)!(n!)} \quad (15)$$
$$= \frac{(n!(n-1)!l_n) \cdot (n!(n-1)!l_n)}{(n-1)!(n!)} \quad (16)$$
$$= \frac{(n!)^2((n-1)!)^2(l_n)^2}{(n-1)!(n!)} \quad (17)$$
$$(18)$$

cancel the same term from the numerator and denominator show that $LS \times RLS = n!(n-1)!(l_n)^2$ this prove that $LS \times RLS = \frac{LS \times LS}{(n-1)!(n!)}$. □

Proposition 3.4. $N_r(LS \times SLS) = \frac{LS \times LS}{(n!)}$

Proof. Since the number of LS is $(n!(n-1)!l_n)^2$ by the theorem 1 and the number of SLS is $(n-1)!l_n$ we know that $LS \times SLS = n!(n-1!l_n)^2$. Moreover we konw that $LS \times LS = (n!(n-1)!l_n)^2$ note following:

$$LS \times SLS = \frac{LS \times LS}{(n!)} \tag{19}$$
$$= \frac{(n!(n-1)!l_n)^2}{(n!)} \tag{20}$$
$$= \frac{(n!(n-1)!l_n)^2}{(n!)} \tag{21}$$
$$= \frac{(n!(n-1)!l_n) \cdot (n!(n-1)!l_n)}{(n!)} \tag{22}$$
$$= \frac{(n!)^2((n-1)!)^2(l_n)^2}{(n!)} \tag{23}$$
$$\tag{24}$$

cancel the same term from the numerator and denominator show that $LS \times SLS = n!(n-1!l_n)^2$. this prove that $LS \times SLS = \frac{LS \times LS}{(n!)}$. □

Proposition 3.5. $N_r(RLS \times RLS) = \frac{LS \times LS}{(n-1!n!)^2}$

Proof. Since the number of RLS is l_n we know that $RLS \times RLS = (l_n)^2$. Moreover we konw that $LS \times LS = (n!(n-1)!l_n)^2$ note following:

$$RLS \times RLS = \frac{LS \times LS}{(n!(n-1)!)^2} \tag{25}$$
$$= \frac{(n!(n-1)!l_n)^2}{(n!(n-1)!)^2} \tag{26}$$
$$= \frac{(n!(n-1)!l_n)^2}{(n!(n-1)!)^2} \tag{27}$$
$$= \frac{(n!(n-1)!l_n) \cdot (n!(n-1)!l_n)}{(n!(n-1)!)^2} \tag{28}$$
$$= \frac{(n!)^2((n-1)!)^2(l_n)^2}{(n!(n-1)!)^2} \tag{29}$$
$$\tag{30}$$

cancel the same term from the numerator and denominator show that $RLS \times RLS = (l_n)^2$ this prove that $RLS \times RLS = \frac{LS \times LS}{(n!(n-1)!)^2}$. □

3.1 Results of r-Orthogonality for Latin Squares of Order 3

Table 2: Number of Superimposition

Sets	Total number of superimposition
$LS \times LS$	144
$SLS \times RLS$	2
$SLS \times SLS$	4
$LS \times RLS$	12
$LS \times SLS$	24

Table 3: r - Orthogonality distribution according to cases

			Set		
$r - orthogonal$	$LS \times LS$	$SLS \times RLS$	$SLS \times SLS$	$LS \times RLS$	$LS \times SLS$
$3 - orthogonal$	72	1	2	6	12
$9 - orthogonal$	72	1	2	6	12

After realizing every possible superimposition of latin squares for order 3, we found that threre are only two different r-orthogonalities: 3-orthogonality and 9-orthogonality. Interestingly enough, there is the same number of 3 and 9 orthogonalities for each case. Knowing the $N(\text{ə})$ in the set $LS \times LS$ we can know the $N(\text{ə})$, also we con know the $N(\text{ə})$ for a special set and the using the propositions 3.1, 3.2, 3.3, 3.4 and 3.5 and applying elemental operations like substitution you will get the $N(LS \times LS)$ a then the $N(\text{ə})$ for the others sets the example 7 show the relations between sets. For this example we do not see the set $RLS \times RLS$ because the LS of order 3 only have one RLS.

Example 7. *Some relation between set using the propositions 3.1, 3.2, 3.3, 3.4 and 3.5.*

1. $N(SLS \times RLS) = \frac{N(LS \times LS)}{2!3!3!}$

2. $N(SLS \times SLS) = \frac{N(LS \times LS)}{3!3!}$
3. $N(LS \times RLS) = \frac{N(LS \times LS)}{2!3!}$
4. $N(LS \times SLS) = \frac{N(LS \times LS)}{3!}$

4 Isotopy Partitions of latin squares

Now we introduce another characteristic of latin squares called Isotopy partitions. **Partitions** are a family of subsets from some set $\mathbb{S}$ denoted φ where φ is not empty ($\varphi \neq \emptyset$), the interseption between partitions is empty ($\varphi_x \cap \varphi_y = \emptyset$) and the union of all the partitions of $\mathbb{S}$ are equal to $\mathbb{S}$ also the elements of one partition are related with a unique relation $\mathbb{R}$ for this partition. The relation $\mathbb{R}$ for the isotopy partition is if you take a LS for the partition φ_x and permute the row, columns and relabels or combination of this permutation the result is a LS in the partition φ_x. Since our goal is study the r-othogonality and know the r - orthogonality with the less number of comparisons between LS we know that is sufficient use the set $RLS \times SLS$ knowing this we decide to study the isotopy partitions to see the distribution of the RLS, SLS also the distribution of the latin square that produce the maximum r - orthogonality. The notation $LS_L^{n,p}$ where n is the order, p is the isotopy partition and L is a label to differentiate the LS, for our convenience the partition representative is denote with $L = 1$ like $LS_1^{n,p}$. We made programs for generate the partitions of latin squares of order 4 and order 5 and investigate the distribution of the RLS, SLS and the LS that produce the maximum r - orthogonality in the isotopy partition.

For the set of latin squares of order 4 and order 5 exist two isotopy partitions that are given by two isotopy partition representatives that are $LS_1^{4,1}$, $LS_1^{4,2}$ and $LS_1^{5,1}$, $LS_1^{5,2}$.

$$LS_1^{4,1} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \qquad LS_1^{4,2} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

$$LS_1^{5,1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \\ 2 & 4 & 3 & 1 & 0 \\ 3 & 0 & 1 & 4 & 2 \\ 4 & 3 & 0 & 2 & 1 \end{pmatrix} \qquad LS_1^{5,2} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 3 & 0 & 4 & 2 \\ 2 & 4 & 3 & 1 & 0 \\ 3 & 0 & 4 & 2 & 1 \\ 4 & 2 & 1 & 0 & 3 \end{pmatrix}$$

Is important to know that other Latin Square in the partition can be one isotopy partition representative. When we refer to the partitions we will use the isotopy partitions representative $LS_1^{n,1}$ or $LS_1^{n,2}$. The distribution of the latin square in each partition is 432 for the isotopy partition $LS_1^{4,1}$, 144 for the isotopy partition $LS_1^{4,2}$, 17280 for the isotopy class $LS_1^{5,1}$ and 144000 for the isotopy class $LS_1^{5,2}$.

4.1 Distributions of the RLS in the Isotopy Partitions

In the case of Latin Squares of order 4 we find that the distribution of the RLS is 1 for the partition $LS_1^{4,1}$ that is the same latin squares that the isotopy partition representative $RLS_1^{4,1}$ and for the partition $LS_1^{4,2}$ we found 3 reduced latin squares that are $RLS_1^{4,2}$, $RLS_2^{4,2}$ and $RLS_3^{4,2}$.

- Reduced Latin Squares in the Patition $LS_1^{4,1}$

$$RLS_1^{4,1} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

- Reduced Latin Squares in the Patition $LS_1^{4,2}$

$$RLS_1^{4,2} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \qquad RLS_2^{4,2} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{pmatrix}$$

$$RLS_3^{4,2} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}$$

The distribution of the reduce latin square for order five is six reduce latin square in the isotopy class $LS_1^{5,1}$ and 50 in the isotopy class $LS_1^{5,2}$. Also in the appendix A we have al the reduce latin square of order 5, the reduce latin square in the isotopy class $LS_1^{5,1}$ are the $14, 23(representative), 24, 38, 40$ and 50, and the other reduce latin square are in the isotopy class $LS_1^{5,2}$.

4.2 Distribution of the LS that produce maximum r - orthogonality in the partition

Studying the results from the isotopy partitions of latin squares of order 4 and order 5 we observe that all the latin squares that produce the maximum r-ortogonality are in the same partition that are $LS_1^{4,2}$ and $LS_1^{5,1}$. This results are given because the maximum r - orthogonality is the absolute maximum, that is that the absolute is n^2. In the proof of the Theorem 2 show that the maximum absolute r - orthogonality is given by the set of mutually orthogonal latin squares also show that the set of mutually orthogonal latin square is made from a RLS and permutations of rows. Moreover this is the reason why all the LS that produce maximum r - orthogonality are in the same isotopy partition. We know this only for the case when the order of the latin square is prime power

Theorem 2. *If q is a prime power, then the size of the largest collection of mutually orthogonal LS of order q is $q-1$.*

This Theorem 2 show that the maximum absolute r - orthogonality exist in all LS of order q when q is a prime power and also show that $\mid MS_L^{n,p} \mid = n-1$. Knowing this we can know all the maximum r - orthogonality in LS of order q and q is a prime power.

Proposition 4.1. *A set of latin square MS of order p^i, p is a prime number and $i \in \mathbb{N}$, the maximum r_t - orthogonality is equal to $(p^i)^2 \cdot \binom{t}{2}$. Let MS be a set of $p^i - 1$ distinct mutually orthogonal latin squares, $MS = \{LS_1, LS_2, \ldots LS_{p^i-1}\}$.*

Proof. Knowing the Theorem 2 is easily to know that all latin squares with order equal to p^i, has a mutually orthogonal latin square. Also the maximum set of mutually orthogonal latin square has $p^i - 1$ latin squares, that is $MS^{p^i} = \{LS_1, LS_2, \ldots, LS_{p^i-1}\}$. If we take two LS from MS^{p^i} we get $(p^i)^2$ - orthogonality moreover note number of possible superimposition of LS in a set MS of p^i-1 latin squares is $\binom{p^i-1}{2}$ knowing this we can say that r_t -orthogonality is equal to $(p^i)^2\cdot\binom{p^i-1}{2}$. $\square$

Example 8. *Use esto si quiere incluir un ejemplo.*

Definition 2. *Use esto si quiere incluir un ejemplo.*

4.3 Escriba aquí el título de la subsección

Contenido de la subsección.

5 Escriba aquí el título de la sección

Contenido de la sección.

6 Apendix A: Reduce Latin Square of order 5 label with the respective isotopy class

$$(1):\ LS_2^5=\begin{pmatrix}01234\\10342\\23410\\34021\\42103\end{pmatrix}(2):\ LS_2^5=\begin{pmatrix}01234\\10342\\23401\\34120\\42013\end{pmatrix}(3):\ LS_2^5=\begin{pmatrix}01234\\10423\\23041\\34102\\42310\end{pmatrix}$$

$$(4):\ LS_2^5=\begin{pmatrix}01234\\10423\\23140\\34012\\42301\end{pmatrix}(5):\ LS_2^5=\begin{pmatrix}01234\\10342\\24013\\32401\\43120\end{pmatrix}(6):\ LS_2^5=\begin{pmatrix}01234\\10342\\24103\\32410\\43021\end{pmatrix}$$

$$(7):\ LS_2^5=\begin{pmatrix}01234\\10423\\24310\\32041\\43102\end{pmatrix}(8):\ LS_2^5=\begin{pmatrix}01234\\10423\\24301\\32140\\43012\end{pmatrix}(9):\ LS_2^5=\begin{pmatrix}01234\\12340\\20413\\34021\\43102\end{pmatrix}$$

$$(10):\ LS_2^5=\begin{pmatrix}01234\\12340\\20413\\34102\\43021\end{pmatrix}(11):\ LS_2^5=\begin{pmatrix}01234\\12403\\20341\\34012\\43120\end{pmatrix}(12):\ LS_2^5=\begin{pmatrix}01234\\12403\\20341\\34120\\43012\end{pmatrix}$$

$$(13):\ LS_2^5=\begin{pmatrix}01234\\12043\\23401\\34120\\40312\end{pmatrix}(14):\ LS_1^5=\begin{pmatrix}01234\\12043\\23410\\34102\\40321\end{pmatrix}(15):\ LS_2^5=\begin{pmatrix}01234\\12340\\23401\\34012\\40123\end{pmatrix}$$

$$(16):\ LS_2^5=\begin{pmatrix}01234\\12403\\23041\\34120\\40312\end{pmatrix}(17):\ LS_2^5=\begin{pmatrix}01234\\12403\\23140\\34012\\40321\end{pmatrix}(18):\ LS_2^5=\begin{pmatrix}01234\\12403\\23140\\34021\\40312\end{pmatrix}$$

$$(19):\ LS_2^5=\begin{pmatrix}01234\\12043\\24301\\30412\\43120\end{pmatrix}(20):\ LS_2^5=\begin{pmatrix}01234\\12043\\24310\\30421\\43102\end{pmatrix}(21):\ LS_2^5=\begin{pmatrix}01234\\12340\\24013\\30421\\43102\end{pmatrix}$$

$$(22):\ LS_2^5=\begin{pmatrix}01234\\12340\\24103\\30412\\43021\end{pmatrix}(23):\ LS_1^5=\begin{pmatrix}01234\\12340\\24103\\30421\\43012\end{pmatrix}(24):\ LS_1^5=\begin{pmatrix}01234\\12403\\24310\\30142\\43021\end{pmatrix}$$

$$(25):\ LS_2^5=\begin{pmatrix}01234\\13042\\20413\\34120\\42301\end{pmatrix}(26):\ LS_2^5=\begin{pmatrix}01234\\13402\\20143\\34021\\42310\end{pmatrix}(27):\ LS_2^5=\begin{pmatrix}01234\\13420\\20143\\34012\\42301\end{pmatrix}$$

$$(28):\ LS_2^5=\begin{pmatrix}01234\\13420\\20341\\34012\\42103\end{pmatrix}(29):\ LS_2^5=\begin{pmatrix}01234\\13402\\20341\\34120\\42013\end{pmatrix}(30):\ LS_2^5=\begin{pmatrix}01234\\13420\\20341\\34102\\42013\end{pmatrix}$$

$$(31):\ LS_2^5=\begin{pmatrix}01234\\13042\\24103\\30421\\42310\end{pmatrix}\ (32):\ LS_2^5=\begin{pmatrix}01234\\13042\\24310\\30421\\42103\end{pmatrix}\ (33):\ LS_2^5=\begin{pmatrix}01234\\13420\\24013\\30142\\42301\end{pmatrix}$$

$$(34):\ LS_2^5=\begin{pmatrix}01234\\13420\\24301\\30142\\42013\end{pmatrix}\ (35):\ LS_2^5=\begin{pmatrix}01234\\13042\\24103\\32410\\40321\end{pmatrix}\ (36):\ LS_2^5=\begin{pmatrix}01234\\13042\\24301\\32410\\40123\end{pmatrix}$$

$$(37):\ LS_2^5=\begin{pmatrix}01234\\13042\\24310\\32401\\40123\end{pmatrix}\ (38):\ LS_2^5=\begin{pmatrix}01234\\13402\\24013\\32140\\40321\end{pmatrix}\ (39):\ LS_1^5=\begin{pmatrix}01234\\13420\\24103\\32041\\40312\end{pmatrix}$$

$$(40):\ LS_2^5=\begin{pmatrix}01234\\13402\\24310\\32041\\40123\end{pmatrix}\ (41):\ LS_1^5=\begin{pmatrix}01234\\14023\\20341\\32410\\43102\end{pmatrix}\ (42):\ LS_2^5=\begin{pmatrix}01234\\14302\\20143\\32410\\43021\end{pmatrix}$$

$$(43):\ LS_2^5=\begin{pmatrix}01234\\14320\\20143\\32401\\43012\end{pmatrix}\ (44):\ LS_2^5=\begin{pmatrix}01234\\14302\\20413\\32041\\43120\end{pmatrix}\ (45):\ LS_2^5=\begin{pmatrix}01234\\14320\\20413\\32041\\43102\end{pmatrix}$$

$$(46):\ LS_2^5=\begin{pmatrix}01234\\14302\\20413\\32140\\43021\end{pmatrix}\ (47):\ LS_2^5=\begin{pmatrix}01234\\14023\\23140\\30412\\42301\end{pmatrix}\ (48):\ LS_2^5=\begin{pmatrix}01234\\14023\\23401\\30142\\42310\end{pmatrix}$$

$$(49):\ LS_2^5=\begin{pmatrix}01234\\14023\\23410\\30142\\42301\end{pmatrix}\ (50):\ LS_2^5=\begin{pmatrix}01234\\14320\\23041\\30412\\42103\end{pmatrix}\ (51):\ LS_1^5=\begin{pmatrix}01234\\14302\\23140\\30421\\42013\end{pmatrix}$$

$$(52):\ LS_2^5 = \begin{pmatrix} 01234 \\ 14320 \\ 23401 \\ 30142 \\ 42013 \end{pmatrix} (53):\ LS_2^5 = \begin{pmatrix} 01234 \\ 14023 \\ 23140 \\ 32401 \\ 40312 \end{pmatrix} (54):\ LS_2^5 = \begin{pmatrix} 01234 \\ 14023 \\ 23401 \\ 32140 \\ 40312 \end{pmatrix}$$

$$(55):\ LS_2^5 = \begin{pmatrix} 01234 \\ 14302 \\ 23041 \\ 32410 \\ 40123 \end{pmatrix} (56):\ LS_2^5 = \begin{pmatrix} 01234 \\ 14302 \\ 23410 \\ 32041 \\ 40123 \end{pmatrix}$$

7 Appendix B: Maximal sets MS_n^5 for latin squares of order 5.

$$MS_1^5 = \left\{ \begin{pmatrix} 01234 \\ 12043 \\ 23410 \\ 34102 \\ 40321 \end{pmatrix}, \begin{pmatrix} 01234 \\ 23410 \\ 12043 \\ 40321 \\ 34102 \end{pmatrix}, \begin{pmatrix} 01234 \\ 40321 \\ 34102 \\ 12043 \\ 23410 \end{pmatrix}, \begin{pmatrix} 01234 \\ 34102 \\ 40321 \\ 23410 \\ 12043 \end{pmatrix} \right\}$$

$$MS_2^5 = \left\{ \begin{pmatrix} 01234 \\ 12340 \\ 24103 \\ 30421 \\ 43012 \end{pmatrix}, \begin{pmatrix} 01234 \\ 24103 \\ 12340 \\ 43012 \\ 30421 \end{pmatrix}, \begin{pmatrix} 01234 \\ 30421 \\ 43012 \\ 24103 \\ 12340 \end{pmatrix}, \begin{pmatrix} 01234 \\ 43012 \\ 30421 \\ 12340 \\ 24103 \end{pmatrix} \right\}$$

$$MS_3^5 = \left\{ \begin{pmatrix} 01234 \\ 12403 \\ 24310 \\ 30142 \\ 43021 \end{pmatrix}, \begin{pmatrix} 01234 \\ 24310 \\ 12403 \\ 43021 \\ 30142 \end{pmatrix}, \begin{pmatrix} 01234 \\ 30142 \\ 43021 \\ 24310 \\ 12403 \end{pmatrix}, \begin{pmatrix} 01234 \\ 43021 \\ 30142 \\ 12403 \\ 24310 \end{pmatrix} \right\}$$

$$MLS_4^5 = \left\{ \begin{pmatrix} 01234 \\ 13402 \\ 24013 \\ 32140 \\ 40321 \end{pmatrix}, \begin{pmatrix} 01234 \\ 24013 \\ 13402 \\ 40321 \\ 32140 \end{pmatrix}, \begin{pmatrix} 01234 \\ 32140 \\ 40321 \\ 24013 \\ 13402 \end{pmatrix}, \begin{pmatrix} 01234 \\ 40321 \\ 32140 \\ 13402 \\ 24013 \end{pmatrix} \right\}$$

$$MS_5^5 = \left\{ \begin{pmatrix} 01234 \\ 13402 \\ 24310 \\ 32041 \\ 40123 \end{pmatrix}, \begin{pmatrix} 01234 \\ 24310 \\ 13402 \\ 40123 \\ 32041 \end{pmatrix}, \begin{pmatrix} 01234 \\ 32041 \\ 40123 \\ 24310 \\ 13402 \end{pmatrix}, \begin{pmatrix} 01234 \\ 40123 \\ 32041 \\ 13402 \\ 24310 \end{pmatrix} \right\}$$

$$MLS_6^5 = \left\{ \begin{pmatrix} 01234 \\ 14320 \\ 23041 \\ 30412 \\ 42103 \end{pmatrix}, \begin{pmatrix} 01234 \\ 23041 \\ 14320 \\ 42103 \\ 30412 \end{pmatrix}, \begin{pmatrix} 01234 \\ 30412 \\ 42103 \\ 23041 \\ 14320 \end{pmatrix}, \begin{pmatrix} 01234 \\ 42103 \\ 30412 \\ 14320 \\ 23041 \end{pmatrix} \right\}$$

References

[1] G. Mullen, & C. Mummert *Finite Fields and Applications*, p. cm - (Student mathematical library; v.41), American Mathematical Society, Rhode Island, 2007.

[2] P. Fitzpatrick , & J. Flynn , *A Gröbner Basis Technique for Padé Approximation*, J. Symbolic Computation, 13 (1992), 133-138.