Study of *r*-Orthogonality of Latin Squares

Jeranfer Bermúdez and Lourdes Morales

University of Puerto Rico – Río Piedras
Department of Computer Science

Mentor: Prof. Ivelisse Rubio

OUTLINE

- Introduction
- Latin Squares
- r–orthogonality
- r_t–orthogonality
- Our Projects
- Summary
- References
- Acknowledgements

LATIN SQUARES

- A **latin square of order *n*** is an n x n matrix containing n distinct symbols such that each symbol appears in each row and column exactly once.

Example.

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

❑ **Theorem.** There is a latin square of order n for each $n \geq 1$.

G. Mullen & C. Mummert, 44.

Why study latin squares?

- Applications
- Puzzles
- It’s fun!

Computational Problem

A central problem in the theory of latin squares is to determine how many latin squares of each size exist.

n	*#LS*
1	1
2	2
3	12
4	576
5	161,280
6	812,851,200
7	61,479,419,904,000
8	108,776,032,459,082,956,800
9	5,524,751,496,156,892,842,531,225,600
10	9,982,437,658,213,039,871,725,064,756,920,320,000
11	776,966,836,171,770,144,107,444,346,734,230,682,311,065,600,000

Taken from N. J. A. Sloane, **A002860**, *On-Line Encyclopedia of Integer Sequences* (1996-2008)
http://www.research.att.com/~njas/sequences/A002860

TYPES OF LATIN SQUARES

- A latin square of order *n* is said to be **reduced** if its first row and first column are in the standard order 0, 1,…, *n*–*1*.

- A latin square of order *n* is said to be **semi-reduced** if its first row is in the standard order.

Example.

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

RLS of order 4

n	*#RLS*
1	1
2	1
3	1
4	4
5	56
6	9,408
7	16,942,080
8	5.35×10^{11}
9	3.78×10^{17}
10	7.58×10^{24}
11	5.36×10^{33}

Taken from N. J. A. Sloane, **A000315**, *On-Line Encyclopedia of Integer Sequences* (1996-2008) http://www.research.att.com/~njas/sequences/A000315.

Theorem. For any $n \geq 2$,

$$\boldsymbol{L_n} = n!\,(n-1)!\,\boldsymbol{l_n}$$

G. Mullen & C. Mummert, 45.

n	*#RLS*
1	1
2	1
3	1
8	5.35×10^{11}
9	3.78×10^{17}
10	7.58×10^{24}
11	5.36×10^{33}

Taken from N. J. A. Sloane, **A000315**, *On-Line Encyclopedia of Integer Sequences* (1996-2008) http://www.research.att.com/~njas/sequences/A000315.

r-ORTHOGONALITY

SUPERIMPOSING LATIN SQUARES

- Given two latin squares of the same size we can **superimpose** them, that is, we can place or lay one latin square over the other to create a square of ordered pairs.
- **Example.**

0	1	2
1	2	0
2	0	1

LS_1

0	1	2
2	0	1
1	2	0

LS_2

(0,0)	(1,1)	(2,2)
(1,2)	(2,0)	(0,1)
(2,1)	(0,2)	(1,0)

$S(LS_1, LS_2)$

r–ORTHOGONALITY

- ***r*** = $P(LS_i, LS_j)$ is the number of distinct ordered pairs you get when you superimpose LS_i and LS_j.

- LS_i and LS_j are said to be ***r*–orthogonal** if ***r*** = $P(LS_i, LS_j)$.

r–ORTHOGONALITY

- **Example.** 8-orthogonal latin squares:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

LS_1

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

LS_2

(0,0)	(1,1)	(2,2)	(3,3)
(1,3)	(2,0)	(3,1)	(0,2)
(2,2)	(3,3)	(0,0)	(1,1)
(3,1)	(0,2)	(1,3)	(2,0)

$S(LS_1,LS_2)$

ORTHOGONAL LATIN SQUARES

- Two latin squares of order n are **orthogonal** if $r = n^2$.

Example:

0	1	2
1	2	0
2	0	1

LS_1

0	1	2
2	0	1
1	2	0

LS_2

(0,0)	(1,1)	(2,2)
(1,2)	(2,0)	(0,1)
(2,1)	(0,2)	(1,0)

$S(LS_1, LS_2)$

MUTUALLY ORTHOGONAL LATIN SQUARES

- A set of **mutually orthogonal latin squares (MOLS)** is a set of two or more latin squares of the same order, all of which are orthogonal to one another.

- **Example.** $\{LS_1, LS_2, LS_3\}$ is mutually orthogonal if $P(LS_1, LS_2)=n^2$, $P(LS_2, LS_3)=n^2$ and $P(LS_1,LS_3)=n^2$.

MUTUALLY ORTHOGONAL LATIN SQUARES

- **Example:**

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

LS_1

0	2	3	1
1	3	2	0
2	0	1	3
3	1	0	2

LS_2

0	3	1	2
1	2	0	3
2	1	3	0
3	0	2	1

LS_3

$P(LS_1, LS_2)=16$ $P(LS_2, LS_3)=16$

$P(LS_1, LS_3)=16$

MUTUALLY ORTHOGONAL LATIN SQUARES

- **Example:** Here we have a set of orthogonal latin

The concept of MOLS of a given order is important, because it is known that there exists a projective plane with *n* points if and only if there are *n – 1* MOLS of order *n*.

$P(LS_1, LS_2)=16$ $P(LS_2, LS_3)=16$

$P(LS_1, LS$

Theorem 2.3.8. (Bose 1938). G. Mullen & C. Mummert, 62.

MUTUALLY ORTHOGONAL LATIN SQUARES

Questions:

Is there a collection of mutually orthogonal latin squares for every order?

If they exist, how big is the largest collection of mutually orthogonal latin squares for each order?

MUTUALLY ORTHOGONAL LATIN SQUARES

- Let $N(n)$ denote the size of the largest collection of *mutually orthogonal latin squares* (MOLS) of order n (that exist).

➢ **Theorem.** $N(n) \geq 2$ for all n except 2 and 6.

$N(2) = 1$ and $N(6) = 1$

➢ **Theorem.** $N(n) \leq n - 1$ for any $n \geq 2$.

➢ **Theorem.** If q is a prime power, then $N(q) = q - 1$.

$q = p^r$, where p is a prime and $r \geq 1$

G. Mullen & C. Mummert.

MUTUALLY ORTHOGONAL LATIN SQUARES

Question:

Are there *n* – 1 mutually orthogonal latin squares of order *n* if *n* is not a prime power?

MUTUALLY ORTHOGONAL LATIN SQUARES

Question:

Are there mutually orthogonal latin

Conjecture. [The Prime Power Conjecture] For $n \geq 2$, $N(n) = n - 1$ if and only if n is a prime power.

G. Mullen & C. Mummert, 48.

r_t-ORTHOGONALITY

r_t–ORTHOGONALITY

- Let $\{LS_1, \ldots, LS_t\}$ be a set of $t \geq 2$ latin squares. Then,

$$r_t = \sum_{i \neq j} P(LS_i, LS_j)$$

- **Example.** Let $\{LS_1, LS_2, LS_3\}$ *:*

$$r_3 = P(LS_1, LS_2) + P(LS_1, LS_3) + P(LS_2, LS_3)$$

r_t–ORTHOGONALITY

- **Example:** $r_3 = 9 + 12 + 9 = 30$

0	1	2	3
1	3	0	2
2	0	3	1
3	2	1	0

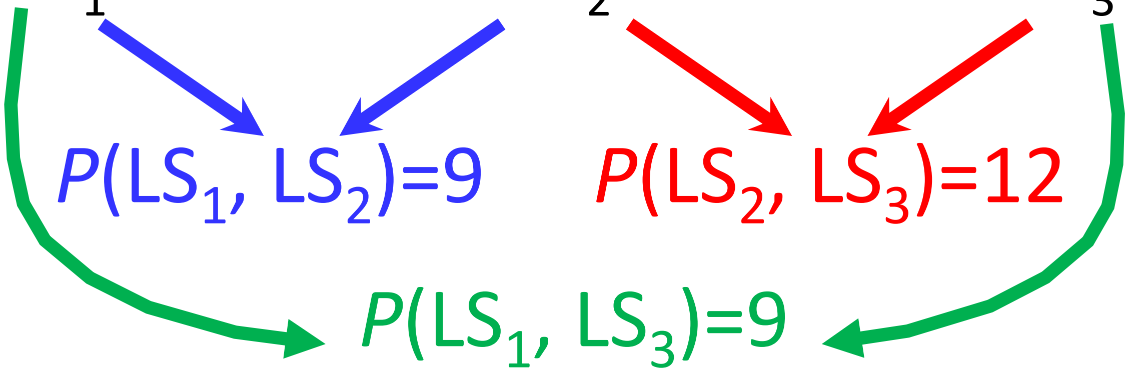
LS_1

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

LS_2

0	1	2	3
2	3	0	1
1	2	3	0
3	0	1	2

LS_3



r_t–ORTHOGONALITY

The r_t-orthogonality is related to how “close” one is from getting projective planes. Thus, computing sets of latin squares of order n that give the maximum r_t-orthogonality, denoted by $\boldsymbol{M_t(n)}$, is an important problem.

$P(LS_1, LS_3)=9$

OUR PROJECTS

Problems

- **What is the maximum r_t–orthogonality, $M_n(t)$?**

What Is Known

For $n=6$ we know these values for the $M_t(6)$

t	$M_t(6)$
2	34
3	96
4	188
5	$300 \leq M_5(6) < 340$

Taken from R. Arce & J. Cordova & I. Rubio. (2009)

Computational Problem

Reduce the number of comparisons and time (the estimated time for computing $M_5(6)$ is 1.95×10^{12} years**)**.

- Restrict focus to special sets of latin squares.
- Eliminate unnecessary comparisons.

Computational Problem

- The plan:
  - –Distribute the work:
    - Cores
    - Processors
    - Computers
  - –Design a specialized circuit that compares latin squares.
  - **–Optimize the algorithm for computing $M_t(6)$**

Problems

- **What is the maximum r_t–orthogonality, $M_n(t)$?**
- **Are there any properties related to $M_n(t)$?**

Problems

- **What is the maximum r_t–orthogonality, $M_n(t)$?**
- **Are there any properties related to $M_n(t)$?**
- **Find constructions for sets of MOLS.**

Sets of MOLS

MOLS Generating Matrix

- Let L be a LS of order n, we say that L is **symmetric** if $L = L^{\mathrm{T}}$.
- **Example**:

$L =$

0	1	2	3	4
1	3	0	4	2
2	0	4	1	3
3	4	1	2	0
4	2	3	0	1

$= L^T$

MOLS Generating Matrix

- A **transversal** in a latin square of order *n* is a set of *n* cells, each from a different row and a different column, such that every element in each cell is different.

- **Example**:
$T = \{L_{1,2} = 1, L_{2,3} = 0, L_{3,5} = 3, L_{4,4} = 2, L_{5,1} = 4\}$
is a transversal in *L*.

$L =$

0	1	2	3	4
1	3	0	4	2
2	0	4	1	3
3	4	1	2	0
4	2	3	0	1

MOLS Generating Matrix

- Setting the value of each cell in the transversal to 1 and the rest of the cells to 0, gives a *permutation matrix*.
- **Example:**

0	1	2	3	4
1	3	0	4	2
2	0	4	1	3
3	4	1	2	0
4	2	3	0	1

0	1	0	0	0
0	0	1	0	0
0	0	0	0	1
0	0	0	1	0
1	0	0	0	0

MOLS Generating Matrix

- Multiplying a permutation matrix by a latin square gives a latin square.
- **Example:**

$$P \times L = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \times \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 \\ \hline 1 & 3 & 0 & 4 & 2 \\ \hline 2 & 0 & 4 & 1 & 3 \\ \hline 3 & 4 & 1 & 2 & 0 \\ \hline 4 & 2 & 3 & 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 0 & 4 & 2 \\ \hline 2 & 0 & 4 & 1 & 3 \\ \hline 4 & 2 & 3 & 0 & 1 \\ \hline 3 & 4 & 1 & 2 & 0 \\ \hline 0 & 1 & 2 & 3 & 4 \\ \hline \end{array} = L_2$$

MOLS Generating Matrix

- Let L be a LS of order $n = q$, where q is a prime power, and let G be an n x n permutation matrix. We say that G is a **MOLS generating matrix** if $\{GL, G^2L, \ldots, G^{n-1}L = L\}$ is a set of MOLS.
- **Example.**

1	3	0	4	2
2	0	4	1	3
4	2	3	0	1
3	4	1	2	0
0	1	2	3	4

2	0	4	1	3
4	2	3	0	1
0	1	2	3	4
3	4	1	2	0
1	3	0	4	2

4	2	3	0	1
0	1	2	3	4
1	3	0	4	2
3	4	1	2	0
2	0	4	1	3

0	1	2	3	4
1	3	0	4	2
2	0	4	1	3
3	4	1	2	0
4	2	3	0	1

MOLS Generating Matrix

- Let L be a LS of order $n = q$, where q is a prime

Conjecture. Let L be a symmetric RLS contained in a set of MOLS. If G is a permutation matrix given by a transversal in L with exactly one 1 on its main diagonal, then G is a MOLS generating matrix.

Bermúdez.

3	4	1	2	0
0	1	2	3	4

3	4	1	2	0
1	3	0	4	2

3	4	1	2	0
2	0	4	1	3

3	4	1	2	0
4	2	3	0	1

SUMMARY

What is Known

- Number of distinct latin squares and distinct reduced latin squares when $n \leq 11$.
- For latin squares of order $n = p^r$, where p is a prime number and $r \geq 1$, $N(n) = n - 1$.
- The maximum r_t–orthogonality when $t \in \{2, 3, 4\}$ and $n = 6$.

What is Unknown

- The number of distinct latin squares and distinct reduced latin squares when $n \geq 12$.
- The maximum r_t–orthogonality when $t = 5$ and $n = 6$.
- The maximum r_t–orthogonality when n is not a prime power and $2 < t \leq n-1$.

FUTURE WORK

- Optimize the computing approach for $M_t(6)$.
- Find properties of the latin squares that produce the $M_t(n)$.
- Find a formula for $M_t(n)$ when n is not a prime power and $t \leq n-1$.
- Modify and prove the MOLS Generating Matrix Conjecture.
- Generalize the concept of the MOLS generating matrix to maximal sets of latin squares.

REFERENCES

- C.J. Colbourn and J.H. Dinitiz, Editors, Handbook of Combinatorial Designs, Sec. Ed., Chapman and Hall/CRC, Boca Raton, FL, 2007.
- F. Castro & C. Corrada & G. Mullen & I. Rubio. (2009). Some Computational Problems for Latin Squares.
- G. Mullen, & C. Mummert. (2007). Finite Fields and Applications, *Chapter 2: Combinatorics. Section 2: Latin Squares*. Pp. 43 – 59. P.cm – Student mathematical library; v.41, American Mathematical Society, Rhode Island, 2007.
- J. Bermúdez. (2009). Study of Latin Square Generating Polynomials.
- N. J. A. Sloane (1996-2008). A002860: Number of Latin squares of order n; or labeled Quasigroups. *On-Line Encyclopedia of Integer Sequences.* Retrieved on February 2, 2009 from *http://www.research.att.com/~njas/sequences/table?a=2860&fmt=4*
- N. J. A. Sloane (1996-2008). A000315: Number of reduced Latin squares of order n; labeled loops (Quasigroups with an identity element) and a fixed identity. *On-Line Encyclopedia of Integer Sequences.* Retrieved on April 4, 2009 from http://www.research.att.com/~njas/sequences/table?a=315&fmt=4
- R. Arce & J. Cordova & I. Rubio. (2008). Consideraciones Computacionales de Latin Squares (Power Point).
- http://cs.anu.edu.au/~bdm/data/latin.html

ACKNOWLEDGEMENTS

This research was done in collaboration with the Latin Square Research Group consisting of:

Prof. Rafael Arce, Prof. Francis Castro, Prof. Javier Córdova, Richard García, Prof. Ivelisse Rubio, University of Puerto Rico – Río Piedras, and Prof. Gary Mullen, Penn State University.

And, with the support of The Puerto Rico Louis Stokes Alliance for Minority Participation (PR-LSAMP) and the National Science Foundation Alan Turing STEM Fellowship.