Abstract

A Latin square (LS) of order n, is an $n \times n$ array of n different elements, where in each row and each column the elements are never repeated. Latin squares have various applications in coding theory, projective geometry and others. Two Latin squares of order $n$ are said to be $r$-orthogonal if when the squares are superimposed we get r distinct ordered pairs of symbols. We study generalizations of the $r$-orthogonality to sets of LS’s. In this work we present preliminary results on some properties of these generalizations.

Introduction

The concept of Latin squares started in 1729 when Euler began working with the “thirty-six officers” problem. The name Latin Square comes from Euler using Latin symbols.

A reduced LS (RLS) has the first row and the first column in the standard order 0, 1, 2, ..., $n-1$.

Example 1: LS$_1$ and LS$_2$ are two reduced Latin squares of order 4 and LS$_3$ is a semi-reduced Latin square of order 4.

\[
\begin{align*}
LS_1 &= \begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
\end{pmatrix}, \\
LS_2 &= \begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
\end{pmatrix}.
\end{align*}
\]

$r$-Orthogonality

Two Latin squares LS$_1$, LS$_2$ are $r$-orthogonal if when the squares are superimposed we get $r = N(LS_1, LS_2)$ distinct ordered pairs of symbols.

Example 2:

\[
S(LS_1, LS_2) = \begin{pmatrix}
(0,0) & (1,1) & (2,2) & (3,3) \\
(1,1) & (2,2) & (3,3) & (0,0) \\
(2,2) & (3,3) & (0,0) & (1,1) \\
(3,3) & (0,0) & (1,1) & (2,2) \\
\end{pmatrix}
\]

Superimposition of LS$_1$ and LS$_2$.

These LS’s are 9-orthogonal because $N(LS_1, LS_2) = 9$.

If $N(LS_1, LS_2) = n^2$ then LS$_1$, LS$_2$ are said to be mutually orthogonal Latin squares (MOLS).

It is known that there exist a projective plane with $n$ points if and only if there are $n-1$ MOLS of order $n$.

Theorem 1: If $q$ is prime power, then the size of the largest collection of MOLS of order $q$ is $q - 1$ and there exists a set of $q - 1$ LS’s MOLS of order $q$.

Problems

1. Find the maximum $r$-orthogonality of sets of $t$ Latin squares.
2. Find constructions for sets of MOLS.

The $r$-orthogonality of sets, measures how “close” one is from getting projective planes. Thus, computing sets with maximal $r$-orthogonality is an important problem.

Let $S = \{LS_1, ... , LS_t\}$ be a set of $t$ LS’s of order $n$, define $r$-orthogonality as

\[
r_{t(n)} = \sum_{i \neq j} N(LS_i, LS_j)
\]

Example 3: Consider the set $S = \{LS_1, LS_2, LS_3\}$ of LS’s from Example 1. Then $N(LS_1, LS_2) = 9$, $N(LS_1, LS_3) = 9$ and $N(LS_2, LS_3) = 12$. Therefore $S$ has $r_{3(4)} = 30$.

The maximum $r$-orthogonality of sets of $t$ LS’s of order $n$ is denoted by $M_{r(n)}$.

Results:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$M_{34}$</th>
<th>$M_{45}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>34</td>
<td>94</td>
</tr>
<tr>
<td>3</td>
<td>34</td>
<td>178</td>
</tr>
<tr>
<td>4</td>
<td>34</td>
<td>188</td>
</tr>
<tr>
<td>5</td>
<td>34</td>
<td>295</td>
</tr>
</tbody>
</table>

Sets of MOLS

Symmetric Reduced Latin Squares

Let $L$ be a reduced Latin square of order $n$, we say that $L$ is a symmetric reduced LS if $L = L^T$.

Example 4: Let $L_1$ be a RLS of order $n = 5$.

\[
L_1 = \begin{pmatrix}
0 & 1 & 234 & 1234 \\
1 & 2 & 341 & 1342 \\
2 & 3 & 412 & 2134 \\
3 & 4 & 123 & 3142 \\
0 & 4 & 231 & 2341 \\
\end{pmatrix}
\]

A transversal of a Latin square of order $n$ is a set of $n$ cells, each from a different row and a different column, such that every element in each cell is different.

Example 5: Consider the Latin square $L_1$, $T$ is a transversal of $L_1$.

\[
T = \{L_{101} = 1, L_{112} = 0, L_{123} = 3, L_{134} = 2, L_{145} = 4\}
\]

Future Work

- Find a formula for $M_{r(n)}$.
- Optimize the algorithm for computing $M_{r(n)}$ because the estimated time for computing $M_{r(n)}$ is 205.52541 years.
- Prove the MOLS Generating Matrix Conjecture.
- Study the relation of the MOLS Generating Matrix with the polynomial that generates the LS.

References

- Mullen, G. & Mummert, C., Finite Fields and Applications, 2007 p. cm - (Student mathematical library; v. 41).

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