Study of r-Orthogonality of Latin Squares

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Abstract

A *Latin square* (*LS*) of order n, is an $n \times n$ array of n different elements, where in each row and each column the elements are never repeated. Latin squares have various applications in coding theory, projective geometry and others. Two Latin squares of order n are said to be *r-orthogonal* if when the squares are superimposed we get r distinct ordered pairs of symbols. We study generalizations of the *r*-orthogonality to sets of *LS's*. In this work we present preliminary results on some properties of these generalizations.

Introduction

The concept of Latin squares started in 1729 when Euler began working with the "thirty-six officers" problem. The name *Latin Square* comes from Euler using Latin symbols.

A *reduced LS* (*RLS*) has the first row and the first column in the standard order *0, 1, 2, ... , n-1*.

Example 1: LS_x and LS_y are two reduced Latin squares of order *4* and LS_z is a semi-reduced Latin square of order *4*.

$$LS_x = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \quad LS_y = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{pmatrix}. \quad LS_z = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 3 & 0 & 1 & 2 \end{pmatrix}$$

r-Orthogonality

Two Latin squares LS_i, LS_j are *r-orthogonal* if when the squares are superimposed we get $r = N(LS_i,LS_j)$ distinct ordered pairs of symbols.

Example 2:

$$S(LS_x, LS_y) = \begin{pmatrix} (0,0) & (1,1) & (2,2) & (3,3) \\ (1,1) & (3,2) & (0,3) & (2,0) \\ (2,2) & (0,3) & (3,0) & (1,1) \\ (3,3) & (2,0) & (1,1) & (0,2) \end{pmatrix}$$

Superimposition of LS_x and LS_y

These *LS's* are 9-orthogonal because $N(LS_x,LS_y) = 9$.

If $N(LS_i,LS_j) = n^2$ then LS_i, LS_j are said to be *mutually orthogonal Latin squares* (*MOLS*).

It is known that there exist a projective plane with n points if and only if there are *n-1 MOLS* of order n.

Theorem 1: If q is prime power, then the size of the largest collection of *MOLS* of order q is $q - 1$ and there exists a set of *q-1 LS's MOLS* of order q.

Problems

1. Find the maximum r_t*-orthogonality* of sets of t Latin squares.
2. Find constructions for sets of *MOLS*.

The *r-orthogonality* of sets, measures how "close" one is from getting projective planes. Thus, computing sets with maximal *r-orthogonality* is an important problem.

Let $S = \{LS_1, ... , LS_t\}$ be a set of t *LS's* of order n, define r_t*-orthogonality* as

$$r_{t(n)} = \sum_{i \neq j} N(LS_i, LS_j)$$

Example 3: Consider the set $S = \{LS_x, LS_y, LS_z\}$ of *LS's* from Example 1. Then $N(LS_x, LS_y) = 9$, $N(LS_x, LS_z) = 9$ and $N(LS_y, LS_z) = 12$. Therefore S has $r_{3(4)} = 30$.

The maximum r_t*-orthogonality* of sets of t *LS's* of order n is denoted by $M_{t(n)}$.

Results:

	t	2	3	4	5
Previous Results	$M_{t(6)}$	34	≥ 94	≥ 178	≥ 295
Our Results	$M_{t(6)}$	34	96	188	≥ 300

Sets of MOLS

Symmetric Reduced Latin Squares

Let L be a reduced Latin square of order n, we say that L is a *symmetric reduced LS* if $L = L^T$.

Example 4: Let L_1 be a RLS of order $n = 5$.

$$L_1 = \begin{pmatrix} 01234 \\ 13042 \\ 20413 \\ 34120 \\ 42301 \end{pmatrix} = L_1^T$$

A *transversal* of a Latin square of order n is a set of n cells, each from a different row and a different column, such that every element in each cell is different.

Example 5: Consider the Latin square L_1. T is a transversal of L_1.

$$T = \{L_{1_{0,1}} = 1, L_{1_{1,2}} = 0, L_{1_{2,4}} = 3, L_{1_{3,3}} = 2, L_{1_{4,0}} = 4\}$$

$$L_1 = \begin{pmatrix} 0\,\boxed{1}\,234 \\ 13\,\boxed{0}\,42 \\ 2041\,\boxed{3} \\ 341\,\boxed{2}\,0 \\ \boxed{4}\,2301 \end{pmatrix}$$

Setting the value of each cell in the transversal to 1 and the rest of the cells to 0, gives a *permutation matrix*. Multiplying a permutation matrix by a *LS* gives a *LS.*

Example 6:

$$G \times L_1 = \begin{pmatrix} 01000 \\ 00100 \\ 00001 \\ 00010 \\ 10000 \end{pmatrix} \times \begin{pmatrix} 01234 \\ 13042 \\ 20413 \\ 34120 \\ 42301 \end{pmatrix} = \begin{pmatrix} 13042 \\ 20413 \\ 42301 \\ 34120 \\ 01234 \end{pmatrix} = L_2$$

MOLS Generating Matrix

Let L_1 be a LS of order $n = q$, *where* q is a prime power, and let G be an *n x n* permutation matrix. We say that G is a *MOLS generating matrix* if $\{GL_1, G^2L_1, ... , G^{n-1}L_1\}$ is a set of MOLS.

Conjecture: Let L_1 be a symmetric *RLS* contained in a set of *MOLS*. If G is a permutation matrix given by a transversal of L_1 with exactly one 1 on its diagonal, then G is a *MOLS generating matrix.*

Example 7: Consider the symmetric reduced Latin square L_1 of order $n = 5$ and the permutation matrix G (from Example 5) given by a L_1 transversal. Then, $\{L_1, GL_1, G^2L_1, G^3L_1\} =$

$$\left\{ \begin{pmatrix} 01234 \\ 13042 \\ 20413 \\ 34120 \\ 42301 \end{pmatrix}, \begin{pmatrix} 13042 \\ 20413 \\ 42301 \\ 34120 \\ 01234 \end{pmatrix}, \begin{pmatrix} 20413 \\ 42301 \\ 01234 \\ 34120 \\ 13042 \end{pmatrix}, \begin{pmatrix} 42301 \\ 01234 \\ 13042 \\ 34120 \\ 20413 \end{pmatrix} \right\}$$

is a set of MOLS, and G is a MOLS generating matrix.

Future Work

- Find a formula for $M_{t(n)}$.
- Optimize the algorithm for computing $M_{t(6)}$ because the estimated time for computing $M_{3(6)}$ is 205.52541 years.
- Prove the MOLS Generating Matrix Conjecture.
- Study the relation of the MOLS Generating Matrix with the polynomial that generates the LS.

References

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